2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

A Convex Approach to Hydrodynamic Analysis

Mohamadreza Ahmadi, Giorgio Valmorbida, Antonis Papachristodoulou

Abstract— **We study stability and input-to-state properties of incompressible, viscous flows with perturbations that are constant in one of the directions. By taking advantage of this flow structure, we propose a class of Lyapunov and storage functionals and consider exponential stability, induced $\mathcal{L}^2$-norms, and input-to-state stability (ISS). For streamwise constant perturbations, we formulate conditions based on matrix inequalities. We show that in the case of polynomial base flow profiles the matrix inequalities can be verified by convex optimization. The proposed method is illustrated by an example of rotating Couette flow.**

I. INTRODUCTION

The dynamics of incompressible fluid flows are described by a set of nonlinear partial differential equations known as the Navier-Stokes equations. The properties of such flows are then characterized in terms of a dimensionless parameter Re called the Reynolds number. Experiments show that many flows have a critical Reynolds number Re_C above which transition to turbulence occurs. However, spectrum analysis of the linearized Navier-Stokes equations, considering only infinitesimal perturbations, predicts a linear stability limit Re_L which upper-bounds Re_C [1]. On the other hand, the bounds using energy methods Re_E, the limiting value for which the energy of arbitrary large perturbations decreases monotonically, are much below Re_C [2]. For example, $Re_E = 32.6$ [3], $Re_L = \infty$ [4] and $Re_C = 350$ [5] for 3D Couette flow.

The discrepancy between Re_L and Re_C have long been attributed to the eigenvalues analysis approach [6], citing a phenomenon called *transient growth*; i.e., although the perturbations to the linearized Navier-Stokes equation are stable, they undergo high amplitude transient amplifications that steer the trajectories out of the region of linearization. This has led to studying the resolvent operator or ε-pseudospectra based on the general solution to the linearized Navier-Stokes equations [7]. Another method for studying stability is based on spectral truncation of the Navier-Stokes equations into an ODE system. Recently in [8], [9], a method was proposed based on keeping a number of modes from Galerkin expansion and bounding the energy of the remaining modes. However, these bounds on Re_C turn out to be conservative.

Since the seminal paper by Reynolds [10], it was observed that external excitations and body forces play an important role in flow instabilities. Mechanisms such as energy amplification of external excitations are important in understanding transition to turbulence [2]. Energy amplification of stochastic forcings to the linearized Navier-Stokes equations in parallel channel flows was studied in [11], [12]. In [12], using the linearized Navier-Stokes equation, it was shown analytically, through the calculation of traces of operator Lyapunov equations, that the $\mathcal{H}^2$-norm from streamwise constant excitations to perturbation velocities is proportional to Re^3. The amplification mechanism of the linearized Navier-Stokes equation was also studied in [13] and [14], where the influence of each component of the body forces was calculated in terms of $\mathcal{H}^2$ and $\mathcal{H}^\infty$-norms.

In this paper, we study the stability and input-to-state properties of incompressible, viscous fluid flows. We study input-to-state properties such as induced $\mathcal{L}^2$-norms from body forces to perturbation velocities and ISS. In particular, we consider flow perturbations that are constant in one of the three spatial coordinates. For such flows, we propose a suitable structure for Lyapunov/storage functionals. Then, based on these functionals, for streamwise constant flows, we derive conditions based on matrix inequalities. In the case of polynomial base velocity profiles and semi-algebraic domains, e.g. Couette and Poiseuille flows, these inequalities can be checked via convex optimization using available computational tools. The proposed method is applied to the rotating Couette flow.

The paper is organized as follows. The next section presents some preliminary results. In Section III, we propose a Lyapunov/storage functional structure. Section IV is concerned with the convex formulation of the analysis problem for streamwise constant flows. The proposed method is illustrated by an example of a model of rotating Couette flow in Section V. Finally, Section VI concludes the paper and provides directions for future research.

Notation: The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The $n \times n$ identity matrix is denoted by $\mathrm{I}_{n\times n}$. A domain $\Omega \subset \mathbb{R}^n$ is a connected, open subset of $\mathbb{R}^n$, and $\overline{\Omega}$ is the closure of set Ω. The boundary $\partial\Omega$ of set Ω is defined as $\overline{\Omega} \setminus \Omega$ with $\setminus$ denoting set subtraction. The space of p-th power integrable functions defined over Ω is denoted $\mathcal{L}^p_\Omega$ endowed with the norm $\|(\cdot)\|_{\mathcal{L}^p_\Omega} = \left(\int_\Omega (\cdot)^p \,\mathrm{d}\Omega\right)^{\frac{1}{p}}$, for $1 \leq p < \infty$, and $\|(\cdot)\|_{\mathcal{L}^\infty_\Omega} = \sup_{x\in\Omega} |(\cdot)|$, for $p = \infty$. Also, we denote by $\mathcal{L}^2_{[t_0,T],\Omega}$, with $t_0 \geq 0$, the space of square integrable functions in $x \in \Omega$ and $t \in [t_0, T]$ with the norm $\|(\cdot)\|_{\mathcal{L}^2_{[t_0,T),\Omega}} = \left(\int_{t_0}^T \|u\|^2_{\mathcal{L}^2_\Omega} \,\mathrm{d}t\right)^{\frac{1}{2}}$. The space of k-times continuous differentiable functions defined

The authors are with the Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK e-mail: ({mohamadreza.ahmadi, giorgio.valmorbida, antonis}@eng.ox.ac.uk). M. Ahmadi is supported by the Clarendon Scholarship and the Sloane-Robinson Scholarship. G. Valmorbida is also affiliated to Somerville College, University of Oxford, Oxford, U.K. A. Papachristodoulou was supported in part by the Engineering and Physical Sciences Research Council projects EP/J012041/1, EP/I031944/1, EP/M002454/1, and EP/J010537/1.

978-1-4799-7886-1/15/$31.00 ©2015 IEEE

on Ω is denoted by $\mathcal{C}^k(\Omega)$. If $p \in \mathcal{C}^1$, then $\partial_{x_1} p$ is used to denote the derivative of p with respect to variable x_1, i.e. $\partial_{x_1} := \frac{\partial}{\partial x_1}$. A continuous strictly increasing function $k : [0, a) \to \mathbb{R}_{\geq 0}$, satisfying $k(0) = 0$, belongs to class $\mathcal{K}$. If $a = \infty$ and $\lim_{x\to\infty} k(x) = \infty$, k belongs to class $\mathcal{K}_\infty$. The unit vector in direction x_i is denoted by $\overrightarrow{e}_i$. For a scalar function v, $\nabla v = \sum_i \partial_{x_i} v \overrightarrow{e}_i$ denotes the gradient and $\nabla^2 v = \sum_i \partial_i^2 v$ denotes the Laplacian. For a vector valued function $\boldsymbol{w} = \sum_i w_i \overrightarrow{e}_i$, the divergence $\nabla \cdot \boldsymbol{w}$ is given by $\nabla \cdot \boldsymbol{w} = \sum_i \partial_{x_i} w_i$.

II. Preliminiaries

A. Flow Model

We consider incompressible, viscous flows with constant perturbations in one of the directions x_m, $m \in \{1, 2, 3\}$, i.e., $\partial_{x_m} = 0$. Let $I = \{1, 2, 3\} - \{m\}$. The flow dynamics are described by the Navier-Stokes equations, given by

$$\begin{aligned} \partial_t \bar{\boldsymbol{u}} &= \frac{1}{Re}\nabla^2 \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} - \nabla \bar{p} + F\bar{\boldsymbol{u}} + \boldsymbol{d}, \\ 0 &= \nabla \cdot \bar{\boldsymbol{u}}, \end{aligned} \tag{1}$$

where $t > 0$, $F \in \mathbb{R}^{3\times 3}$ accounts for terms added due to rotation, and $\mathrm{x} \in \Omega = \Omega_i \times \Omega_j \subset \mathbb{R} \times \mathbb{R}$ with $\mathrm{x} = (x_i, x_j)'$, $i, j \in I$, $i \neq j$, being the spatial coordinates. The dependent variable $\boldsymbol{d}(t, \mathrm{x}) = \begin{bmatrix} d_1(t, \mathrm{x}) & d_2(t, \mathrm{x}) & d_3(t, \mathrm{x}) \end{bmatrix}'$ is the input vector representing exogenous excitations or body forces, $\bar{\boldsymbol{u}}(t, \mathrm{x}) = \begin{bmatrix} \bar{u}_1(t, \mathrm{x}) & \bar{u}_2(t, \mathrm{x}) & \bar{u}_3(t, \mathrm{x}) \end{bmatrix}'$ is the velocity vector, and $\bar{p}(t, \mathrm{x})$ is the pressure.

We consider perturbations $(\boldsymbol{u}, p)$ to the stationary flow $(\boldsymbol{U}, P)$. That is,

$$\bar{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{U}, \ \bar{p} = p + P, \tag{2}$$

where $(\boldsymbol{U}, P)$ satisfy

$$\begin{aligned} 0 &= \frac{1}{Re}\nabla^2 \boldsymbol{U} - \boldsymbol{U} \cdot \nabla \boldsymbol{U} - \nabla P + F\boldsymbol{U}, \\ 0 &= \nabla \cdot \boldsymbol{U}. \end{aligned} \tag{3}$$

Substituting (2) in (1) and using (3), we obtain the perturbation dynamics

$$\begin{aligned} \partial_t \boldsymbol{u} &= \frac{1}{Re}\nabla^2 \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{U} \\ & \quad -\nabla p + F\boldsymbol{u} + \boldsymbol{d}, \\ 0 &= \nabla \cdot \boldsymbol{u}. \end{aligned} \tag{4}$$

In this paper, we concentrate on perturbations with no-slip boundary conditions $\boldsymbol{u}|_{\partial\Omega} \equiv 0$ and periodic boundary conditions.

B. Stability and Input-to-State Analysis

In this section, we briefly review a number of definitions and results from [15], [16] (also see the 2D formulation in [17]) and [18], [19].

Definition 1 (Exponential Stability): The stationary solution $(0, p_0)$ of (4) with $\boldsymbol{d} \equiv 0$ is exponentially stable in $\mathcal{L}^2_\Omega$, if there exists a $\lambda > 0$, such that for all $t \geq 0$

$$\|\boldsymbol{u}(t, \mathrm{x})\|^2_{\mathcal{L}^2_\Omega} \leq \|\boldsymbol{u}(0, \mathrm{x})\|^2_{\mathcal{L}^2_\Omega} e^{-\lambda t}. \tag{5}$$

That is, system (1) converges to the base flow $(\boldsymbol{U}, P)$ satisfying (3).

Definition 2 (input-to-State Properties):

A. *Induced $\mathcal{L}^2$-norm Boundedness*: For some $\eta_i > 0$, $i = 1, 2, 3$,

$$\|\boldsymbol{u}(t, \mathrm{x})\|^2_{\mathcal{L}^2_{[0,\infty),\Omega}} \leq \sum_{i=1}^{3} \eta_i^2 \|d_i(t, \mathrm{x})\|^2_{\mathcal{L}^2_{[0,\infty),\Omega}} \tag{6}$$

subject to zero initial conditions $\boldsymbol{u}(0, \mathrm{x}) \equiv 0$, $\forall \mathrm{x} \in \Omega$.

B. *Input-to-State Stability*: For some scalar $\psi > 0$, functions $\beta, \tilde{\beta}, \chi \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{K}$, it holds that

$$\begin{aligned} \|\boldsymbol{u}(t, \mathrm{x})\|_{\mathcal{L}^2_\Omega} &\leq \beta\Big(e^{-\psi t}\chi\left(\|\boldsymbol{u}(0, \mathrm{x})\|_{\mathcal{L}^2_\Omega}\right)\Big) \\ &+ \tilde{\beta}\bigg(\sup_{\tau\in[0,t)} \Big(\int_\Omega \sigma\big(|\boldsymbol{d}(\tau, \mathrm{x})|\big)\, \mathrm{d}\Omega\Big)\bigg), \end{aligned} \tag{7}$$

for all $t > 0$.

Remark 1: Due to nonlinear dynamics, the actual induced $\mathcal{L}^2$-norms of system (4) are nonlinear functions of $\|\boldsymbol{d}\|_{\mathcal{L}^2_{[0,\infty),\Omega}}$. The quantities η_i, $i = 1, 2, 3$ provide upper-bounds on the actual induced $\mathcal{L}^2$-norms.

Remark 2: The ISS property (7) implies the exponential convergence to the base flow $(\boldsymbol{U}, P)$ in $\mathcal{L}^2_\Omega$ when $\boldsymbol{d} \equiv 0$. Moreover, as $t \to \infty$, we obtain

$$\begin{aligned} \lim_{t\to\infty} \|\boldsymbol{u}(t, \mathrm{x})\|_{\mathcal{L}^2_\Omega} &\leq \beta\left(\int_\Omega \|\sigma(|\boldsymbol{d}(t, \mathrm{x})|)\|_{\mathcal{L}^\infty_{[0,\infty)}}\, \mathrm{d}\Omega\right) \\ &\leq \beta\left(\int_\Omega \sigma(\|\boldsymbol{d}(t, \mathrm{x})\|_{\mathcal{L}^\infty_{[0,\infty)}})\, \mathrm{d}\Omega\right), \end{aligned} \tag{8}$$

using the fact that $\sigma, \beta \in \mathcal{K}$. Hence, as long as the external excitations or body forces $\boldsymbol{d}$ are bounded in $\mathcal{L}^\infty_{[0,\infty)}$ (this encompasses persistent excitations), the perturbation velocities $\boldsymbol{u}$ are bounded in the $\mathcal{L}^2_\Omega$ sense.

The next result converts the tests for exponential stability, induced $\mathcal{L}^2$-norm boundedness, and ISS into the existence of a Lyapunov or a storage functional satisfying a set of inequalities.

Theorem 1: Consider perturbation model (4). If there exist a positive definite Lyapunov functional $V(\boldsymbol{u})$ and a positive semidefinite storage functional $S(\boldsymbol{u})$, positive scalars $\{\eta_i\}_{i\in\{1,2,3\}}$, $\{c_i\}_{i\in\{1,2,3\}}$, ψ, and functions $\beta_1, \beta_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, such that

I) when $\boldsymbol{d} \equiv 0$,

$$c_1\|\boldsymbol{u}\|^2_{\mathcal{L}^2_\Omega} \leq V(\boldsymbol{u}) \leq c_2\|\boldsymbol{u}\|^2_{\mathcal{L}^2_\Omega}, \tag{9}$$

$$\partial_t V(\boldsymbol{u}) \leq -c_3\|\boldsymbol{u}\|^2_{\mathcal{L}^2_\Omega}, \tag{10}$$

II)

$$\partial_t S(\boldsymbol{u}) \leq -\int_\Omega \boldsymbol{u}'\boldsymbol{u}\, \mathrm{d}\Omega + \int_\Omega \boldsymbol{d}' \begin{bmatrix} \eta_1^2 & 0 & 0 \\ 0 & \eta_2^2 & 0 \\ 0 & 0 & \eta_3^2 \end{bmatrix} \boldsymbol{d}\, \mathrm{d}\Omega, \tag{11}$$

III)

$$\beta_1(\|\boldsymbol{u}\|_{\mathcal{L}^2_\Omega}) \leq S(\boldsymbol{u}) \leq \beta_2(\|\boldsymbol{u}\|_{\mathcal{L}^2_\Omega}), \tag{12}$$

$$\partial_t S(\boldsymbol{u}) \leq -\psi S(\boldsymbol{u}) + \int_\Omega \sigma(|\boldsymbol{d}(t, \mathrm{x})|)\, \mathrm{d}\Omega, \tag{13}$$

for all $t > 0$, then, respectively, system (4)
I) is exponentially stable,
II) has induced $\mathcal{L}^2$-*norm* upper-bounds η_i, $i = 1, 2, 3$ as in (6),
III) is *ISS* and satisfies (7) with $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1} \circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1} \circ \frac{2}{\psi}(\cdot)$.

Proof: This is a direct application of Theorem 1 in [15] and Theorem 1 in [18]. ■

III. Lyapunov and Storage Functionals for Fluid Flows

In this section, we derive classes of Lyapunov and storage functionals suitable for analysis of system (4) subject to invariance in one of the three spatial coordinates. In the following, we adopt Einstein's multi-index notation over index j, that is the sum over repeated indices j, e.g., $v_j \partial_{x_j} u_j = \sum_j v_j \partial_{x_j} u_j$.

The perturbation model (4) can be re-written as

$$\begin{aligned} \partial_t u_i &= \frac{1}{Re}\nabla^2 u_i - u_j \partial_{x_j} u_i - U_j \partial_{x_j} u_i \\ &\quad -u_j \partial_{x_j} U_i - \partial_{x_i} p + F_{ij} u_j + d_i, \\ 0 &= \partial_{x_j} u_j. \end{aligned} \quad (14)$$

where $i, j \in \{1, 2, 3\}$ and F_{ij} is the (i, j) entry of F.

The next proposition presents a Lyapunov/storage functional structure for which the time derivative of the Lyapunov/storage functional takes the form of a quadratic form in the dependent variables $\boldsymbol{u}$ and their spatial derivatives, by removing the nonlinear convection and pressure terms.

Proposition 1: Consider the perturbation model (14) subject to periodic or no-slip boundary conditions $\boldsymbol{u}|_{\partial\Omega} = 0$. Assume constant perturbations with respect to x_m, $m \in \{1, 2, 3\}$. Let $I = \{1, 2, 3\} - \{m\}$ and

$$\begin{aligned} V(\boldsymbol{u}) &= \frac{1}{2}\int_\Omega \boldsymbol{u}' \begin{bmatrix} k_m & 0 & 0 \\ 0 & k_i & 0 \\ 0 & 0 & k_j \end{bmatrix} \boldsymbol{u}\, \mathrm{d}\Omega \\ &= \frac{1}{2}\int_\Omega \sum_{i=1}^{3} k_i u_i(t, \mathrm{x})^2\, \mathrm{d}\Omega, \end{aligned} \quad (15)$$

where $k_i = k_j$ for $i, j \in I$, be a candidate Lyapunov or storage functional. Then, the time derivative of (15) along the solutions of (14) satisfies

$$\begin{aligned} \partial_t V(\boldsymbol{u}) \le -\sum_{i=1}^{3} k_i \int_\Omega \Bigg(&\frac{C(\Omega)}{Re} u_i^2 + U_j u_i \partial_{x_j} u_i \\ &+ u_j u_i \partial_{x_j} U_i - u_i F_{ij} u_j \Bigg)\, \mathrm{d}\Omega, \end{aligned} \quad (16)$$

where $C(\Omega) > 0$ is the Poincaré constant.

The proof follows from computing the time derivative of Lyapunov/storage functional (15) along the solutions of (14) and applying integration-by-parts and boundary conditions. The proof is omitted here due to space restrictions.

Remark 3: A special case of (15) was used in [20] to study the stability of viscous fluid flows.

Remark 4: In the sequel, we use structure (15) as a Lyapunov functional when studying stability and as a storage functional when studying input-to-state properties.

Remark 5: There are several estimates for the optimal Poincaré constant $C(\Omega)$. The (optimal) constant used here is $C(\Omega) = \frac{\pi^2}{D(\Omega)}$, where $D(\Omega)$ is the diameter of the domain Ω [21].

The next corollary proposes conditions under which input-to-state properties can be inferred for the flow described by (14).

Corollary 1: Consider the perturbation dynamics described by (14) subject to periodic or no-slip boundary conditions $\boldsymbol{u}|_{\partial\Omega} = 0$. Assume that the perturbations are constant with respect to x_m, $m \in \{1, 2, 3\}$. Let $I = \{1, 2, 3\} - \{m\}$. If there exist positive constants k_i, $i = 1, 2, 3$, with $k_i = k_j$, $i, j \in I$, positive scalars $\{\psi_i\}_{i\in\{1,2,3\}}$, $\{\eta\}_{i\in\{1,2,3\}}$, and $\sigma \in \mathcal{K}$ such that
I) when $\boldsymbol{d} \equiv 0$ and $\boldsymbol{u} \neq 0$,

$$\begin{aligned} \sum_{i=1}^{3} k_i \int_\Omega \Bigg(&\frac{C(\Omega)}{Re} u_i^2 + U_j u_i \partial_{x_j} u_i \\ &+ u_j u_i \partial_{x_j} U_i - u_i F_{ij} u_j \Bigg)\, \mathrm{d}\Omega > 0 \end{aligned} \quad (17)$$

II)

$$\begin{aligned} \sum_{i=1}^{3} \int_\Omega \Bigg(&\left(\frac{k_i C(\Omega)}{Re} - 1 \right) u_i^2 + k_i U_j u_i \partial_{x_j} u_i \\ &+ k_i u_j u_i \partial_{x_j} U_i - k_i u_i F_{ij} u_j - k_i u_i d_i + \eta_i^2 d_i^2 \Bigg)\, \mathrm{d}\Omega \ge 0 \end{aligned} \quad (18)$$

III)

$$\begin{aligned} \sum_{i=1}^{3} \int_\Omega \Bigg(&\left(\frac{k_i C(\Omega)}{Re} - \psi_i k_i \right) u_i^2 + k_i U_j u_i \partial_{x_j} u_i \\ &+ k_i u_j u_i \partial_{x_j} U_i - k_i u_i F_{ij} u_j \\ &- k_i u_i d_i + \sigma(|d_1|, |d_2|, |d_3|) \Bigg)\, \mathrm{d}\Omega \ge 0 \end{aligned} \quad (19)$$

then,
I) perturbation velocities given by (14) are exponentially stable. Therefore, the flow converges to the base flow exponentially.
II) under zero perturbation initial conditions $\boldsymbol{u}(0, \mathrm{x}) \equiv 0$, the induced $\mathcal{L}^2$ norm from inputs to perturbation velocities is bounded by η_i, $i \in \{1, 2, 3\}$ as in (6).
III) the perturbation velocities described by (14) are ISS in the sense of (7).

Proof: The proof is omitted. ■

IV. Convex Formulation for Streamwise Constant Flows

To convexify the conditions in Corollary 1, we restrict our attention to streamwise constant perturbations in x_m-direction with base flow $\boldsymbol{U} = U_m(\mathrm{x})\overrightarrow{e}_m$.

Corollary 2: Consider the perturbation dynamics given by (14). Assume streamwise constant perturbations in the x_m-direction with base flow $\boldsymbol{U} = U_m(\mathrm{x})\overrightarrow{e}_m$ where $m \in \{1, 2, 3\}$. Let $I = \{1, 2, 3\} - \{m\}$. If there exist positive

$$M(\mathrm{x}) = \begin{bmatrix} \left(\frac{C}{Re} - F_{mm}\right) k_m & \frac{k_m(\partial_{x_j} U_m(\mathrm{x}) - F_{mj}) - k_j F jm}{2} & \frac{k_m(\partial_{x_i} U_m(\mathrm{x}) - F_{mi}) - k_i F im}{2} \\ \frac{k_m(\partial_{x_j} U_m(\mathrm{x}) - F_{mj}) - k_j F jm}{2} & \left(\frac{C}{Re} - F_{jj}\right) k_j & -\frac{k_j F_{jm}}{2} \\ \frac{k_m(\partial_{x_i} U_m(\mathrm{x}) - F_{mi}) - k_i F im}{2} & -\frac{k_j F_{jm}}{2} & \left(\frac{C}{Re} - F_{ii}\right) k_i \end{bmatrix} \geq 0, \;\; i,j \in I, i \neq j, \;\; \mathrm{x} \in \Omega. \tag{20}$$

constants $\{k_l\}_{l\in\{1,2,3\}}$ with $k_p = k_q$, $p, q \in I$, $\{\eta_l\}_{l\in\{1,2,3\}}$, $\{\psi_l\}_{l\in\{1,2,3\}}$, and functions $\{\sigma_l\}_{l\in\{1,2,3\}}$ such that
I) (20) holds
II)

$$N(\mathrm{x}) = \left[\begin{array}{ccc:ccc} & & & -\frac{k_m}{2} & 0 & 0 \\ & M(\mathrm{x}) - \mathrm{I}_{3\times 3} & & 0 & -\frac{k_j}{2} & 0 \\ & & & 0 & 0 & -\frac{k_i}{2} \\ \hdashline -\frac{k_m}{2} & 0 & 0 & \eta_m^2 & 0 & 0 \\ 0 & -\frac{k_j}{2} & 0 & 0 & \eta_i^2 & 0 \\ 0 & 0 & -\frac{k_i}{2} & 0 & 0 & \eta_j^2 \end{array}\right] \geq 0 \tag{21}$$

for $i, j \in I, i \neq j$ and $\mathrm{x} \in \Omega$,
III) $\sigma_l(\mathrm{x}) \geq 0, \;\; \mathrm{x} \in \Omega$, $l \in \{1,2,3\}$ and

$$P(\mathrm{x}) = \left[\begin{array}{ccc:ccc} & & & -\frac{k_m}{2} & 0 & 0 \\ & M(\mathrm{x}) - Q & & 0 & -\frac{k_j}{2} & 0 \\ & & & 0 & 0 & -\frac{k_i}{2} \\ \hdashline -\frac{k_m}{2} & 0 & 0 & \sigma_m(\mathrm{x}) & 0 & 0 \\ 0 & -\frac{k_j}{2} & 0 & 0 & \sigma_j(\mathrm{x}) & 0 \\ 0 & 0 & -\frac{k_i}{2} & 0 & 0 & \sigma_i(\mathrm{x}) \end{array}\right] \geq 0 \tag{22}$$

for $i, j \in I, i \neq j$ and $\mathrm{x} \in \Omega$, where $Q = \begin{bmatrix} \psi_m k_m & 0 & 0 \\ 0 & \psi_j k_j & 0 \\ 0 & 0 & \psi_i k_i \end{bmatrix}$, then
I) the perturbation velocities are exponentially stable,
II) subject to zero initial conditions, the induced $\mathcal{L}^2$ norm from inputs to perturbation velocities is bounded by η_i, $i = 1, 2, 3$ as in (6),
III) the perturbation velocities are ISS in the sense of (7) with $\sigma(|\boldsymbol{d}|) = \sum_{i=1}^{3} \sigma_i(\mathrm{x}) d_i^2$.

Proof: The proof is omitted for brevity. ■

In the case that $U_m(\mathrm{x})$ is polynomial in x, inequalities (20), (21), and (22) are polynomial matrix inequalities that should be checked for all $\mathrm{x} \in \Omega$. If the set Ω is a semi-algebraic set then these inequalities can be cast as a sum-of-squares (SOS) program by using Putinar's Positivstellensatz [22, Theorem 2.14], which can be solved via semi-definite programming [23], [24], [25] using software packages such as SOSTOOLS [26] .

Remark 6: In order to find upper-bounds on the induced $\mathcal{L}^2$-norm from the body forces (d_1, d_2, d_3) to the perturbation velocities $\boldsymbol{u}$, we solve the following optimization problem

$$\min_{\{k_i\}_{i\in I}} (\eta_1^2 + \eta_2^2 + \eta_3^2)$$
$$\text{subject to } N(\mathrm{x}) \geq 0, \;\; k_i > 0, \;\; i \in I. \tag{23}$$

In the next section, we consider the analysis of the rotating Couette flow, which illustrates the proposed results.

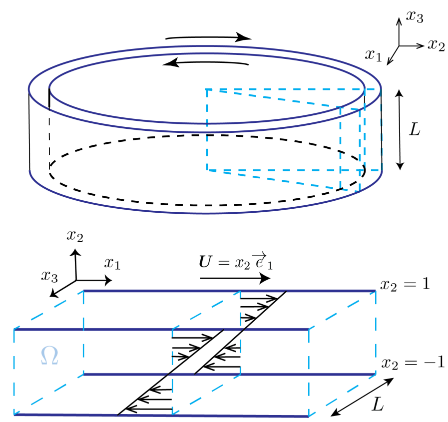

Fig. 1: Schematic of the rotating Couette flow geometry.

V. Example: Rotating Couette Flow

We consider the flow of viscous fluid between two co-axial cylinders, where the gap between the cylinders is much smaller than their radii. In this setting, the flow can be schematically illustrated as in Figure 1. The axis of rotation is parallel to x_3-axis and the circumferential direction corresponds to x_1-axis. Then, the dynamics of the perturbation velocities is described by (4). The perturbations are assumed to be constant with respect to x_1 ($\partial_{x_1} = 0$) and periodic in x_3 with period L. Therefore, $\Omega = \{(x_2, x_3) \mid (x_2, x_3) \in [-1, 1] \times [0, L]\}$. The base flow is given by $\boldsymbol{U} = (x_2, 0, 0)' = x_2 \vec{e}_1$ and $P = P_0$. In addition, $F = \begin{bmatrix} 0 & Ro & 0 \\ -Ro & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $Ro \in [0, 1]$ is a parameter representing the Coriolis force[1]. We consider no-slip boundary conditions $\boldsymbol{u}|_{x_2=-1}^{1} = 0$ and $\boldsymbol{u}(t, x_2, x_3) = \boldsymbol{u}(t, x_2, x_3 + L)$. The Poincaré constant is then given by $C = \frac{\pi^2}{L^2 + 2^2}$.

Notice that the cases $Ro = 0, 1$ correspond to the Couette flow. Thus, the obtained results for rotating Couette flow can be applied to the Couette flow in special cases, as well. We are interested in finding estimates of the critical Reynolds

[1]That is, $Ro = 0$ ($Ro = 1$) corresponds to the case where only the outer (inner) cylinder is rotating and $Ro = 0.5$ is the case where both cylinders are rotating with the same velocity but in opposite direction.

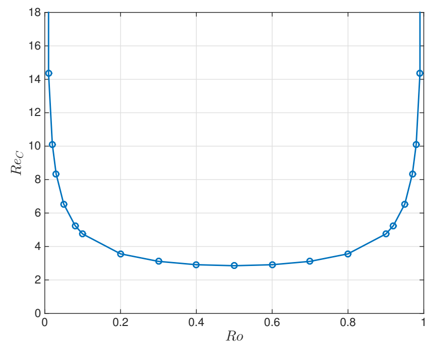

Fig. 2: Estimated critical Reynolds numbers Re in terms of Ro for rotating Couette flow.

number Re_C using the following Lyapunov functional

$$V(u)=\int_0^L\int_{-1}^1 \begin{bmatrix} u_1\\ u_2\\ u_3\end{bmatrix}'\begin{bmatrix} k_1 & 0 & 0\\ 0 & k_2 & 0\\ 0 & 0 & k_2\end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ u_3\end{bmatrix}\ \mathrm{d}x_2\mathrm{d}x_3,$$

which is the same as Lyapunov functional (15) assuming invariance with respect to x_1.

For stability analysis, we need to check inequality (20) according to Item I in Corollary 2. For this flow ($m=1, j=2, i=3$), we have

$$M=\begin{bmatrix} \frac{k_1C}{Re} & \frac{k_2Ro-k_1(Ro-1)}{2} & 0\\ \frac{k_2Ro-k_1(Ro-1)}{2} & \frac{k_2C}{Re} & 0\\ 0 & 0 & \frac{k_2C}{Re}\end{bmatrix}\geq 0. \quad (24)$$

This is a linear matrix inequality (LMI) feasibility problem with decision variables $k_1, k_2>0$.

To find estimates of Re_C in the case of Couette Flow $Ro=0$, applying Schur complement theorem [27, p. 650] to (24), we have $\frac{k_1C}{Re}-\left(\frac{k_1}{2}\right)^2\left(\frac{Re}{k_2C}\right)\geq 0$ and $\frac{k_2C}{Re}\geq 0$, which yields[2] $\frac{k_2}{k_1}\geq\left(\frac{Re}{2C}\right)^2$. This implies that the Couette flow is stable for all Re. Hence, for Couette flow, $Re_C=\infty$ obtained using Lyapunov functional (15) coincides with linear stability limit $Re_L=\infty$ [4].

Let $L=\pi$. Figure 2 illustrates the estimated critical Reynolds numbers Re_C as a function of Ro obtained from solving the LMI (24) and performing a line search over Re. Notice that for the cases $Ro=0,1$ the flow is stable for all Reynolds numbers.

For induced $\mathcal{L}^2$-norm analysis, we apply inequality (21) which for this particular flow is given by the following LMI

$$N=\left[\begin{array}{ccc:ccc} & & & -\frac{k_1}{2} & 0 & 0\\ & M-\mathrm{I}_{3\times 3} & & 0 & -\frac{k_2}{2} & 0\\ & & & 0 & 0 & -\frac{k_2}{2}\\ \hdashline -\frac{k_1}{2} & 0 & 0 & \eta_1^2 & 0 & 0\\ 0 & -\frac{k_2}{2} & 0 & 0 & \eta_2^2 & 0\\ 0 & 0 & -\frac{k_2}{2} & 0 & 0 & \eta_3^2\end{array}\right]\geq 0$$

[2]For $Ro=1$, we can similarly obtain $\frac{k_1}{k_2}\geq\left(\frac{Re}{2C}\right)^2$.

with M as in (24).

Figure 3 depicts the obtained results for three different Reynolds numbers. As the Reynolds number approaches the estimated Re_C for $Ro=0.5$, the upper-bounds on the induced $\mathcal{L}^2$-norm from the body forces $\boldsymbol{d}$ to perturbation velocities $\boldsymbol{u}$ increase dramatically.

The obtained upper-bounds on the induced $\mathcal{L}^2$-norm for Couette flow $Ro=0$, are also given in Figure 4. The obtained upper-bounds depicted in Figure 4 imply $\eta_1^2=c_0Re+c_1Re^{2.3}$, $\eta_2^2=f_0Re^2+f_1Re^4$ and $\eta_3^2=g_0Re^2+g_1Re^4$ with $c_0,c_1,f_0,f_1>0$. This can compared with Theorem 11 in [14, p. 11], wherein the authors calculated componentwise $\mathcal{H}^\infty$-norms for the linearized 2D/3C model by finding the maximum singular values, i.e., $\eta_1^2\propto O(Re^2)$, and $\eta_2^2,\eta_3^2\propto O(Re^4)$ for Couette flow.

In order to check the ISS property, we check inequality (22) from Corollary 2 for the rotating Couette flow under study, i.e.,

$$P=\left[\begin{array}{ccc:ccc} & & & -\frac{k_1}{2} & 0 & 0\\ & M-Q & & 0 & -\frac{k_2}{2} & 0\\ & & & 0 & 0 & -\frac{k_2}{2}\\ \hdashline -\frac{k_1}{2} & 0 & 0 & \sigma_1 & 0 & 0\\ 0 & -\frac{k_2}{2} & 0 & 0 & \sigma_2 & 0\\ 0 & 0 & -\frac{k_2}{2} & 0 & 0 & \sigma_3\end{array}\right]\geq 0$$

with M given in (24) and $Q=\begin{bmatrix} k_1\psi_1 & 0 & 0\\ 0 & k_2\psi_2 & 0\\ 0 & 0 & k_2\psi_3\end{bmatrix}$. We fix $\psi_i=10^{-4}$, $i=1,2,3$ and $L=2\pi$. Numerical experiments show that, for $Ro\in(0,1)$, the maximum Reynolds number for which ISS certificates could be found Re_{ISS} and Re_C coincide. However, for the case of Couette flow $Ro=0,1$, we obtain $Re_{ISS}=316$ and $Re_C=\infty$.

VI. CONCLUSIONS AND FUTURE WORK

We studied stability and input-to-state properties of fluid flows subject to constant perturbations in one of the directions. We formulated a class of appropriate Lyapunov/storage functionals for such flows. Conditions based on matrix inequalities are given for streamwise constant flows, that can be checked using convex optimization for polynomial base flows. For illustration purposes, we applied the proposed method to study a model of rotating Couette flow.

In this study, we considered flows in the Cartesian coordinate system. For many flows, like Hagen-Poiseuille flow, the coordinate system is naturally cylindrical. An extension of the results proposed in this paper to cylindrical coordinates is under study. Moreover, in several scenarios in fluid mechanics, we are interested in a functional of the perturbation dynamics. For example, in the drag estimation problem, we are interested in estimating the functional of pressure over the surface of an airfoil. We are currently applying the methodology proposed in [28] to address such problems in fluid mechanics.

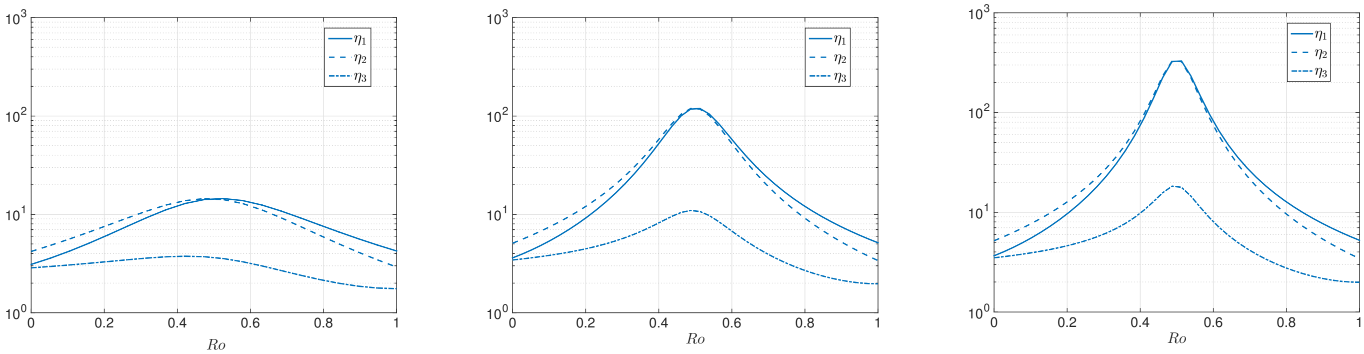

Fig. 3: Upper bounds on induced $\mathcal{L}^2$-norms from $\boldsymbol{d}$ to perturbation velocities $\boldsymbol{u}$ of rotating Couette flow for different Reynolds numbers: $Re = 2$ (left), $Re = 2.8$ (middle), and $Re = 2.83$ (right).

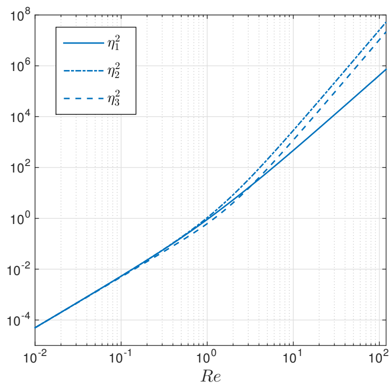

Fig. 4: Upper bounds on induced $\mathcal{L}^2$-norms for perturbation velocities of Couette flow for different Reynolds numbers.

REFERENCES

[1] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*. New York: Cambridge University Press, 1981.

[2] D. D. Joseph, *Stability of fluid motions*, ser. Springer Tracts in Natural Philosophy. Berlin: Springer-Verlag, 1976.

[3] J. Serrin, "On the stability of viscous fluid motions," *Arch. Ration. Mech. Anal.*, vol. 3, pp. 1–13, 1959.

[4] V. Romanov, "Stability of plane-parallel Couette flow," *Functional Analysis and Its Applications*, vol. 7, no. 2, pp. 137–146, 1973.

[5] N. Tillmark and P. H. Alfredsson, "Experiments on transition in plane Couette flow," *Journal of Fluid Mechanics*, vol. 235, pp. 89–102, 1992.

[6] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, "Hydrodynamic stability without eigenvalues," *Science*, vol. 261, no. 5121, pp. 578–584, 1993.

[7] P. J. Schmid, "Nonmodal stability theory," *Annual Review of Fluid Mechanics*, vol. 39, no. 1, pp. 129–162, 2007.

[8] P. J. Goulart and S. Chernyshenko, "Global stability analysis of fluid flows using sum-of-squares," *Physica D: Nonlinear Phenomena*, vol. 241, no. 6, pp. 692 – 704, 2012.

[9] S. Chernyshenko, P. Goulart, D. Huang, and A. Papachristodoulou, "Polynomial sum of squares in fluid dynamics: a review with a look ahead," *Royal Society of London. Philosophical Transactions A. Mathematical, Physical and Engineering Sciences*, vol. 372, no. 2020, 2014.

[10] O. Reynolds, "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and the law of resistance in parallel channels," *Philos. Trans.*, vol. 935, no. 51, 1883.

[11] B. F. Farrell and P. J. Ioannou, "Stochastic forcing of the linearized Navier-Stokes equations," *Physics of Fluids A*, vol. 5, no. 11, pp. 2600–2609, 1993.

[12] B. Bamieh and M. Dahleh, "Energy amplification in channel flows with stochastic excitation," *Physics of Fluids*, vol. 13, no. 11, pp. 3258–3269, 2001.

[13] M. R. Jovanović and B. Bamieh, "Componentwise energy amplification in channel flows," *Journal of Fluid Mechanics*, vol. 534, pp. 145–183, 2005.

[14] M. R. Jovanović, "Modeling, analysis, and control of spatially distributed systems," Ph.D. dissertation, University of California, Santa Barbara, 2004.

[15] G. Valmorbida, M. Ahmadi, and A. Papachristodoulou, "Semi-definite programming and functional inequalities for distributed parameter systems," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, Dec 2014, pp. 4304–4309.

[16] ——, "Stability analysis for a class of partial differential equations via semi-definite programming," *IEEE Transactions on Automatic Control*, May 2016, in Press.

[17] ——, "Convex solutions to integral inequalities in two dimensions," in *Decision and Control (CDC), 2015 IEEE 54rd Annual Conference on*, Osaka, Japan, Dec 2015.

[18] M. Ahmadi, G. Valmorbida, and A. Papachristodoulou, "Input-output analysis of distributed parameter systems using convex optimization," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, Dec 2014, pp. 4310–4315.

[19] ——, "Dissipation inequalities for the analysis of a class of PDEs," *Automatica*, 2016, conditionally Accepted.

[20] D. D. Joseph and W. Hung, "Contributions to the nonlinear theory of stability of viscous flow in pipes and between rotating cylinders," *Archive for Rational Mechanics and Analysis*, vol. 44, no. 1, pp. 1–22, 1971.

[21] L. Payne and H. Weinberger, "An optimal Poincare inequality for convex domains," *Archive for Rational Mechanics and Analysis*, vol. 5, no. 1, pp. 286–292, 1960.

[22] J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*. Imperial College Press, London, 2009.

[23] M. Choi, T. Lam, and B. Reznick, "Sums of squares of real polynomials," in *Symposia in Pure Mathematics*, vol. 58, no. 2, 1995, pp. 103–126.

[24] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, 2000.

[25] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *5th European Control Conference*, Karlsruhe, Germany, 1999.

[26] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, `http://arxiv.org/abs/1310.4716`, 2013, available from `http://www.eng.ox.ac.uk/control/sostools`.

[27] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[28] M. Ahmadi, G. Valmorbida, and A. Papachristodoulou, "Barrier functionals for output functional estimation of PDEs," in *2015 American Control Conference*, July 2015.