H_∞ Stabilization of Uncertain Piecewise Linear Systems with Filippov Solutions

Mohamadreza Ahmadi * **Hamed Mojallali** *
Rafael Wisniewski **

* *Department of Electrical Engineering, Faculty of Engineering, University of Guilan, Rasht, Iran, P.O. Box: 41635-3756, Postal Code: 41996-13769, Tel: +98 131 6690270, Fax: +98 131 6690271, (email: mrezaahmadi@aol.com; mojallali@guilan.ac.ir).*
** *Department of Electronic Systems, Automation and Control, Aalborg University, Fredrik Bajers Vej 7 C, 9220, Aalborg East, Denmark, Tel: +45 9940 8762, Fax: +45 9815 1739, (email: raf@es.aau.dk).*

Abstract: This paper is concerned with robust stabilization and H_∞ control of piecewise linear switched systems subject to uncertainty in the context of Filippov solutions. The analysis developed here encapsulates solutions with infinite switching in finite time. First, a set of linear matrix inequalities are brought forward which determines sufficient conditions to investigate the asymptotic stability of Filippov solutions. Then, a method to construct robust stabilizing switching controllers is proposed. Subsequently, the latter result is extended to switching controllers with H_∞ performance. The suggested controller synthesis schemes are based on solving sets of bilinear matrix inequalities for which appropriate algorithms are delineated.

Keywords: Switched Systems, Filippov Solutions, Robust Control, Bilinear Matrix Inequalities

1. INTRODUCTION

An important class of hybrid systems are piecewise linear (PWL) systems, which has received tremendous attention in open literature (de Best et al. (2008), Azuma et al. (2008), Dobrescu et al. (2008) Johansson and Rantzer (1998), Goncalves et al. (2003), Chan et al. (2004), Ohta and Yokoyama (2010), Leth and Wisniewski (2012), Johansson (2003), Sun (2010), Branicky (1998), Hassibi and Boyd (1998),Dobrescu et al. (2008)). By a PWL system, we understand a family of linear systems defined on polyhedral sets such that the dynamics inside a polytope is governed by a linear dynamic equation. The union of these polyhedral sets forms the state-space. We say that a "switch" has occurred whenever a trajectory passes to an adjacent polytope.

The stability analysis of PWL systems is an intricate assignment. It is established that even if all the subsystems are stable, the overall system may possess divergent trajectories (Branicky (1998)). Furthermore, the behavior of solutions along the boundary of polytopes (facets) may engender unstable trajectories where transitions are, generally speaking, multi-valued. That is, a PWL system with stable Carathéodory solutions may possess divergent Filippov solutions such that the overall system is unstable (see Example 5 in Leth and Wisniewski (2012)). Hence, the stability of the Carathédory solutions does not imply the stability of the overall PWL system.

The stability problem of PWL systems has been addressed by a number of researchers (Johansson and Rantzer (1998), Leth and Wisniewski (2012)). An efficacious contribution was made by Johansson and Rantzer (1998). The authors proposed a number of LMI feasibility tests to investigate the exponential stability of a given PWL system by introducing the concept of piecewise quadratic Lyapunov functions. Following the same trend, Chan et al. (2004) extended the results to the case of uncertain PWL systems. The authors also brought forward a H_∞ controller synthesis scheme for uncertain PWL systems based on a set of LMI conditions.

However, the solutions considered implicitly in both contributions are defined in the sense of Carathéodory. This means that a solution of a PWL system is the concatenation of classical solutions on the facets of polyhedral sets. This connotates that sliding phenomena or solutions with infinite switching in finite time are disregarded which according to (Leth and Wisniewski (2012)) is a critical drawback. Motivated by recent trends in discontinuous control systems (Boiko (2009)) and the popular sliding mode control techniques (Edwards et al. (2006)), in this study we consider Filippov solutions (Filippov (1988)) instead of the conventional Caratheódory solutions. Not to mention that this subsumes sliding modes and solutions with infinite switching in finite time. Our approach is established upon the results reported by Leth and Wisniewski (2012), wherein the authors applied the theory of differential inclusions to derive stability theorems for switched systems with Filippov solutions. The results expounded in this paper are formulated as a set of LMI or bilinear matrix inequality (BMI) conditions which can be cast as semi-definite programming problems.

The framework of this paper is organized as follows. The subsequent section discusses the robust stability results. In Section 3, a stabilizing state-feedback controller for uncertain PWL systems is formulated. The H_∞ Controller synthesis methodology and a V-K iteration algorithm to deal with the BMI conditions are described in Section 4. The accuracy of the proposed method is evaluated by two simulation examples in Section 5. The paper ends with conclusions in Section 6.

2. ROBUST STABILITY

2.1 Notations and Definitions

Consider a class of PWL systems with Filippov solutions $\mathcal{S} = \{\mathcal{X}, \mathcal{U}, \mathcal{V}, X, I, F, G\}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ is a polyhedral set representing the state space, $\mathcal{X} = \{X_i\}_{i\in I}$ is the set containing the polytopes in $\mathcal{X}$ with index set $I = \{1, 2, \ldots, n_X\}$ (note that $\bigcup_{i\in I} X_i = \mathcal{X}$). Each polytope X_i is characterized by the set $\{x \in \mathcal{X} \mid E_i x \succcurlyeq 0\}$ where the notation $\succcurlyeq$ signifies the component-wise inequality. $\mathcal{U}$ is the control space and $\mathcal{V}$ is the disturbance space, which are both subsets of Euclidean spaces. In addition, each function $v(t)$ belongs to $\mathcal{L}_2[0,\infty)$. $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ are families of linear functions associated with the system states x and outputs y. Each f_i consists of six elements $(A_i, B_i, D_i; \Delta A_i, \Delta B_i, \Delta D_i)$ and each g_i is composed of four elements $(C_i, G_i; \Delta C_i, \Delta G_i)$. Furthermore, $f_i : Y_i \times \mathcal{U} \times \mathcal{V} \to \mathbb{R}^n; (x, u, v) \mapsto \big\{z \in \mathbb{R}^n \mid z = (A_i + \Delta A_i)x + (B_i + \Delta B_i)u + (D_i + \Delta D_i)v\big\}$ and $g_i : Y_i \times \mathcal{U} \to \mathbb{R}^m; (x, u) \mapsto \big\{z \in \mathbb{R}^m \mid z = (C_i + \Delta C_i)x + (G_i + \Delta G_i)u\big\}$ where Y_i is an open neighborhood of X_i. The set of matrices $(A_i, B_i, C_i, D_i, G_i)$ are defined over the polytope X_i and $(\Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta G_i)$ encompass the corresponding uncertainty terms. The dynamics of the system can be described by

$$\dot{x}(t) \in co\Big(\mathcal{F}\big(x(t), u(t), v(t)\big)\Big) \tag{1}$$

$$y(t) \in \mathcal{G}\big(x(t), u(t)\big) \tag{2}$$

where, $co(\cdot)$ denotes the convex hull, the set valued maps (Aubin and Cellina (1984)) $\mathcal{F}$ and $\mathcal{G}$ are defined as

$$\begin{aligned}\mathcal{F} : \mathcal{X} \times \mathcal{U} \times \mathcal{V} &\to 2^{\mathcal{X}}\\ ; (x, u, v) &\mapsto \{z \in \mathbb{R}^n \mid z = f_i(x, u, v) \text{ if } x \in X_i\}\end{aligned} \tag{3}$$

$$\begin{aligned}\mathcal{G} : \mathcal{X} \times \mathcal{U} &\to 2^{\mathbb{R}^m}\\ ; (x, u) &\mapsto \{z \in \mathbb{R}^m \mid z = g_i(x, u) \text{ if } x \in X_i\}\end{aligned} \tag{4}$$

where the notation 2^A means the power set or the set of all subsets of A. Denote by $\tilde{I} = \{(i, j) \in I^2 \mid X_i \cap X_j \neq \emptyset,\ i \neq j\}$ the set of index pairs which determines the polytopes with non-empty intersections. We now assume that each polytope is the intersection of a finite set of supporting half spaces. By N_{ij} denote the normal vector pertained to the hyperplane supporting both X_i and X_j. Consequently, each boundary can be characterized as

$$X_i \cap X_j = \{x \in \mathcal{X} \mid N_{ij}^T x \approx 0,\ H_{ij} x \succcurlyeq 0,\ (i, j) \in \tilde{I}\} \tag{5}$$

where $\approx$ represent the component-wise equality and the inequality $H_{ij} x \succcurlyeq 0$ confines the hyperplane to the interested region. Throughout the paper, the matrix inequalities should be understood in the sense of positive definiteness; i.e., $A > B$ $(A \geq B)$ means $A-B$ is positive definite (semi-definite). In case of matrix inequalities, I denotes the unity matrix (the size of I can be inferred from the context) and should be distinguished from the index set I. In matrices, $\star$ in place of a matrix entry a_{mn} means that $a_{mn} = a_{nm}^T$.

A Filippov solution to (1) is an absolutely continuous function $[0, T) \to \mathcal{X}; t \mapsto \phi(t)$ $(T > 0)$ which solves the following Cauchy problem

$$\dot{\phi}(t) \in co\Big(\mathcal{F}\big(\phi(t), u(t), v(t)\big)\Big) \quad \text{a.e.,} \quad \phi(0) = \phi_0 \tag{6}$$

In the sequel, it is assumed that at any interior point $x \in \mathcal{X}$ there exists a Filippov solution to system (1). This can be evidenced by Proposition 5 in (Leth and Wisniewski (2012)). For more information pertaining to the solutions and their existence or uniqueness properties, the interested reader is referred to the expository review (Cortes (1998)) and the didactic book (Filippov (1988)). We underscore that there exists plenty of definitions for solutions of discontinuous systems e.g. Krasovskii, Aizerman and Gantmakher (see Cortes (1998) and Chapter 2 in Yakubovich et al. (2004) for a comparison); however, only the Filippov solutions enjoy a rich theoretical developement made by A.F. Filippov (Filippov (1988)) and have been extensively exploited in the analysis of discontinuous systems encountered in engineering applications (Giaouris et al. (2008), Paden and Sastry (1987), Jing et al. (2011), Forti and Nistri (2003)).

2.2 Stability of PWL systems with Filippov Solutions

In (Leth and Wisniewski (2012)), a stability theorem for switched systems with Filippov solutions is proposed which is reformulated for PWL systems in the next proposition.

Proposition 1. Consider the following autonomous PWL system

$$\dot{x} \in co\big(\mathcal{F}(x)\big) \tag{7}$$

with $\Delta A_i \approx 0$. If there exists quadratic forms $\Phi_i(x) = x^T Q_i x$, $\Psi_i(x) = x^T(A_i^T Q_i + Q_i A_i)x$ and $\Psi_{ij}(x) = x^T(A_j^T Q_i + Q_i A_j)x$ satisfying

$$\Phi_i(x) > 0 \quad \text{for all} \quad x \in X_i \setminus \{0\} \tag{8}$$

$$\Psi_i(x) < 0 \quad \text{for all} \quad x \in X_i \setminus \{0\} \tag{9}$$

for all $i \in I$, and

$$\Psi_{ij}(x) < 0 \quad \text{for all} \quad x \in X_i \cap X_j \setminus \{0\} \tag{10}$$

$$\Phi_i(x) = \Phi_j(x) \quad \text{for all} \quad x \in X_i \cap X_j \tag{11}$$

for all $(i, j) \in \tilde{I}$. Then, the the equilibrium point 0 of (7) is asymptotically stable.

Remark 1. The inclusions $x \in X_i \setminus \{0\}$ and $x \in X_i \cap X_j$ are analogous to $\{x \in \mathcal{X} \mid E_i x \succ 0\}$ and (5), respectively.

It is worth noting that Conditions (8)-(9) are concerned with the positivity of a quadratic form over a polytope; whereas, (10) is about positivity over a hyperplane. Condition (11) asserts that the candidate Lyapunov functions should be continuous (along the facets). A well known LMI formulation of conditions (8), (9) and (11) was proposed in (Johansson and Rantzer (1998)) which is described next. Let us construct a set of matrices F_i, $i \in I$ such that $F_i x = F_j x$ for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$. Then, it follows that the piecewise linear candidate Lyapunov functions can be formulated as

$$V(x) = x^T F_i^T M F_i x = x^T Q_i x \quad \text{if} \quad x \in X_i \tag{12}$$

where, the free parameters of Lyapunov functions are concentrated in the symmetric matrix M. In the following proposition we generalize the results proposed by Johansson and Rantzer (1998) to PWL systems with the more general Filippov solutions.

Proposition 2. Consider the PWL system (7) with Fillipov solutions, and the family of piecewise quadratic Lyapunov functions $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$. If there exist a set of symmetric matrices Q_i, three sets of symmetric matrices U_i, S_i, T_{ij} with non-negative entries, and matrices W_{ij} of appropriate dimensions with $i \in I$ and $(i,j) \in \tilde{I}$, such that the following LMI problem is feasible

$$Q_i - E_i^T S_i E_i > 0 \tag{13}$$

$$A_i^T Q_i + Q_i A_i + E_i^T U_i E_i < 0 \tag{14}$$

for all $i \in I$, and

$$A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} < 0 \tag{15}$$

for all $(i,j) \in \tilde{I}$. Then, the equilibrium point 0 of (7) is asymptotically stable.

Proof. Matrix inequalities (13) and (14) are the same as Equation (11) in Theorem 1 in (Johansson and Rantzer (1998)) which satisfy (8)-(9). The continuity of the Lyapunov functions is also ensured from the assumption that $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$ since $F_i x = F_j x$, for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$. (10) is equivalent to $x^T(A_j^T Q_i + Q_i A_j)x < 0$ for $\{x \in \mathcal{X} \mid N_{ij}^T x \approx 0,\ H_i j x \succ 0\}$. Applying the S-procedure and Finsler's lemma (Polik and Terlaky (2007)), we obtain (15) for a set of matrices T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries and W_{ij}, $(i,j) \in \tilde{I}$ with appropriate dimensions.

We remark that algorithms for constructing matrices E_i and F_i, $i \in I$, are described in (Johansson (2003)).

Remark 2. A similar LMI formulation to (11) can be found in (Johansson (2003)); whereas, our analysis, in this paper, is established upon the stability theorem delineated in Proposition 10 in (Leth and Wisniewski (2012)) which considered the Filippov Solutions.

2.3 Uncertain PWL Systems with Filippov Solutions

Henceforth, we will focus on the family of uncertain PWL systems given by (1). In order to derive the stability and control results, we assume that the upper bound of uncertainties are known apriori; i.e.,

$$\begin{aligned} \Delta A_i{}^T \Delta A_i &\leq \mathcal{A}_i^T \mathcal{A}_i \\ \Delta B_i{}^T \Delta B_i &\leq \mathcal{B}_i^T \mathcal{B}_i \\ \Delta C_i{}^T \Delta C_i &\leq \mathcal{C}_i^T \mathcal{C}_i \\ \Delta D_i{}^T \Delta D_i &\leq \mathcal{D}_i^T \mathcal{D}_i \\ \Delta G_i{}^T \Delta G_i &\leq \mathcal{G}_i^T \mathcal{G}_i \end{aligned} \tag{16}$$

in which, $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i, \mathcal{G}_i)$ are any set of constant matrices with the same dimension as $(A_i, B_i, C_i, D_i, G_i)$ satisfying (16).

Lemma 1. Consider the uncertain PWL system (7). If there exist small positive constants ϵ_i, $i \in I$, ϵ_{ij}, $(i,j) \in \tilde{I}$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$, with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$, of appropriate dimensions, such that

$$Q_i - E_i^T S_i E_i > 0 \tag{17}$$

$$\begin{bmatrix} \Xi_i & Q_i \\ \star & -\epsilon_i \mathrm{I} \end{bmatrix} < 0 \tag{18}$$

for all $i \in I$, and

$$\begin{bmatrix} \Xi_{ij} & Q_i \\ \star & -\epsilon_{ij} \mathrm{I} \end{bmatrix} < 0 \tag{19}$$

for all $(i,j) \in \tilde{I}$, where $\Xi_i = A_i^T Q_i + Q_i A_i + E_i^T U_i E_i + \epsilon_i \mathcal{A}_i^T \mathcal{A}_i$ and $\Xi_{ij} = A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \epsilon_{ij} \mathcal{A}_j^T \mathcal{A}_j$. Then, every Filippov solution of the autonomous uncertain system (7) converges to the origin asymptotically.

Proof. Condition (17) is equivalent to (13). We need to show that (18) and (19) correspond to (14) and (15), respectively. Substituting the uncertain vector $\bar{A}_i = A_i + \Delta A_i$ in (15) yields $(A_j + \Delta A_j)^T Q_i + Q_i(A_j + \Delta A_j) + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} < 0$ which with little manipulation leads to $A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \Delta A_j^T Q_i + Q_i \Delta A_j \leq A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \frac{1}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij} \mathcal{A}_j^T \mathcal{A}_j$. Using Shur complement theorem, we arrive at (19). The equivalency of (18) to (14) can also be proved in a similar manner.

Remark 3. Notice that if the conditions (17)–(19) hold, then (7) is also asymptotically stable for any ΔA_i satisfying (16).

3. STABILIZING STATE FEEDBACK CONTROLLER DESIGN

We are interested in designing a switching controller

$$\begin{aligned} &u \in \mathcal{K}(x) \\ \mathcal{K} : \mathcal{X} \to 2^{\mathcal{U}}; x \mapsto &\left\{ z \in \mathcal{U} \mid z = K_i x \text{ if } x \in X_i \right\} \end{aligned} \tag{20}$$

for system (1) such that all Filippov solutions of (1) denoted by $\phi(t)$ satisfy $\lim_{t \to \infty} \phi(t) = 0$. Considering a controller with the structure given by (20), the controlled system with $v \approx 0$ reduces to (7) with $\mathcal{F}$ supplanted by $\acute{\mathcal{F}} : \mathcal{X} \to 2^{\mathcal{X}}; x \mapsto \{z \in \mathcal{X} \mid z = A_{ci} x \text{ if } x \in X_i\}$, wherein $A_{ci} = A_i + \Delta A_i + (B_i + \Delta B_i) K_i$.

Lemma 2. The controlled switched system as defined above is asymptotically stable at the origin provided that there exist small positive constants ϵ_i, $i \in I$, ϵ_{ij}, $(i,j) \in \tilde{I}$, matrices K_i, $i \in I$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries, matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^T S_i E_i > 0 \tag{21}$$

$$\begin{bmatrix} \Xi_i & Q_i & K_i^T B_i^T & K_i^T \mathcal{B}_i^T \\ \star & \frac{-\epsilon_i}{3+\epsilon_i^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_i}{1+\epsilon_i^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_i}\mathrm{I} \end{bmatrix} < 0 \tag{22}$$

for all $i \in I$, and

$$\begin{bmatrix} \Xi_{ij} & Q_i & K_j^T B_j^T & K_j^T \mathcal{B}_j^T \\ \star & \frac{-\epsilon_{ij}}{3+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_{ij}}\mathrm{I} \end{bmatrix} < 0 \tag{23}$$

for all $(i,j) \in \tilde{I}$.

Proof. We need to demonstrate that (22) and (23) correspond to (14) and (15), respectively. Substituting A_{ci} in (15) yields $A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \Delta A_j^T Q_i + Q_i \Delta A_j + K_j^T B_j^T Q_i + Q_i B_j K_j + K_j^T \Delta B_j^T Q_i + Q_i \Delta B_j K_j \leq A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \frac{1}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij} \Delta A_j^T \Delta A_j + \frac{1}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij} K_j^T \Delta B_j^T \Delta B_j K_j + K_j^T B_j^T Q_i + Q_i B_j K_j \leq \Xi_{ij} + (\epsilon_{ij} + \frac{3}{\epsilon_{ij}}) Q_i Q_i + (\epsilon_{ij} + \frac{1}{\epsilon_{ij}}) K_j^T B_j^T B_j K_j + \epsilon_{ij} K_j^T \mathcal{B}_j^T \mathcal{B}_j K_j$. Utilizing Shur complement theorem, we derive (23). Derivation of (22) can be done similarly.

Remark 4. The conditions derived in Lemma 2 are BMIs (Antwerp and Braatz (2000)) in the variables Q_i and K_i.

To surmount the BMI tests given in Lemma 2, the following V-K iteration algorithm is suggested:

- *Initialization*: Select a set of controller gains based on pole placement method or any other controller design scheme to predetermine a set of initial controller gains.
- *Step V*: Given the set of fixed controller gains K_i, $i \in I$, solve the following optimization problem

$$\min_{Q_i, S_i, U_i, T_{ij}} \gamma_i$$
$$\text{subject to (21) and}$$

$$\begin{bmatrix} \Xi_i & Q_i & K_i^T B_i^T & K_i^T \mathcal{B}_i^T \\ \star & \frac{-\epsilon_i}{3+\epsilon_i^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_i}{1+\epsilon_i^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_i}\mathrm{I} \end{bmatrix} - \gamma_i \mathrm{I} < 0, \tag{24}$$

$$\begin{bmatrix} \Xi_{ij} & Q_i & K_j^T B_j^T & K_j^T \mathcal{B}_j^T \\ \star & \frac{-\epsilon_{ij}}{3+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_{ij}}\mathrm{I} \end{bmatrix} - \gamma_i \mathrm{I} < 0 \tag{25}$$

  for a set of matrices Q_i, $i \in I$.
- *Step K*: Given the set of fixed controller gains Q_i, $i \in I$, solve the following optimization problem

$$\min_{K_i, S_i, U_i, T_{ij}} \gamma_i$$
$$\text{subject to (21), (24), and (25)}$$

  for a set of matrices K_i, $i \in I$.

The algorithm continues till $\gamma_i < 0$, $i \in I$.

4. ROBUST CONTROLLER SYNTHESIS WITH H_∞ PERFORMANCE

In this section, we propose a set of conditions to design a stabilizing switching controller of the form (20) with a guaranteed H_∞ performance (Dullerud and Paganini (2000), Stoica (2005), Serbanescu and Popeea (2004)). That is, a controller such that, in addition to asymptotic stability, ensures that the induced $\mathcal{L}_2$-norm of the operator from $v(t)$ to the controller output $y(t)$ is less than a constant $\eta > 0$ under zero initial conditions ($x(0) = 0$); in other words,

$$\frac{1}{2}\left(\int_0^\infty y^T(\tau) y(\tau) \mathrm{d}\tau\right)^{\frac{1}{2}} \leq \frac{\eta}{2}\left(\int_0^\infty v^T(\tau) v(\tau) \mathrm{d}\tau\right)^{\frac{1}{2}} \tag{26}$$

given any non-zero $v \in \mathcal{L}_2[0,\infty)$.

If we apply the switching controller (20) to (1)-(2), we arrive at the following controlled system with outputs

$$\begin{aligned} \dot{x}(t) &\in co\Big(\tilde{\mathcal{F}}\big(x(t), v(t)\big)\Big) \\ y(t) &\in \tilde{\mathcal{G}}\big(x(t)\big) \end{aligned} \tag{27}$$

where, $\tilde{\mathcal{F}} : \mathcal{X} \times \mathcal{V} \to 2^{\mathcal{X}}; (x,v) \mapsto \{z \in \mathbb{R}^n \mid z = A_{ci}x + D_{ci}v \text{ if } x \in X_i\}$ and $\mathcal{G} : \mathcal{X} \to 2^{\mathbb{R}^m}; x \mapsto \{z \in \mathbb{R}^m \mid z = C_{ci}(x) \text{ if } x \in X_i\}$ with

$$\begin{aligned} A_{ci} &= A_i + \Delta A_i + (B_i + \Delta B_i) K_i \\ D_{ci} &= D_i + \Delta D_i \\ C_{ci} &= C_i + \Delta C_i + (G_i + \Delta G_i) K_i \end{aligned} \tag{28}$$

Proposition 3. System (27) is asymptotically stable at the origin with disturbance attenuation η as defined in (26), if there exist a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^T S_i E_i > 0 \tag{29}$$

$$A_{ci}^T Q_i + Q_i A_{ci} + E_i^T U_i E_i + \eta^{-2} Q_i D_{ci} D_{ci}^T Q_i + C_{ci}^T C_{ci} < 0 \tag{30}$$

for all $i \in I$, and

$$\begin{aligned} A_{cj}^T Q_i + Q_i A_{cj} + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} \\ + \eta^{-2} Q_i D_{cj} D_{cj}^T Q_i + C_{cj}^T C_{cj} < 0 \end{aligned} \tag{31}$$

for all $(i,j) \in \tilde{I}$.

Proof. From (29)-(30) and Proposition 2, it can be discerned that the Filippov solutions of the closed loop system (27) converge to origin asymptotically. Additionally, since $Q_i = F_i^T M F_i$ and $F_i x = F_j x$, for all $x \in X_i \cap X_j$ the continuity of the Lyapunov functions is assured. What remains is to show that the disturbance attenuation performance is η. Define a multi-valued function

$$\Gamma(x) = \{z \in \mathbb{R} \mid z = V_i(x), \quad \text{if} \quad x \in X_i\} \tag{32}$$

This can be thought of as a switched Lyapunov function. Differentiating and integrating Γ with respect to t yields

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}\Gamma}{\mathrm{d}t}\mathrm{d}t = & \int_0^{t_1} \big[x^T(A_{c1}^T Q_1 + Q_1 A_{c1})x \\ & + v^T D_{c1}^T Q_1 x + x^T Q_1 D_{c1} v\big]\mathrm{d}t + \dots \\ & + \int_{t_1}^{t_2} \big[x^T(A_{c2}^T Q_2 + Q_2 A_{c2})x \\ & + v^T D_{c2}^T Q_2 x + x^T Q_2 D_{c2} v\big]\mathrm{d}t + \dots \\ & + \sum_{j=1}^{r} \alpha_j \bigg\{ \int_{t_{k-1}}^{t_k} \big[x^T(A_{cj}^T Q_k + Q_k A_{cj})x \\ & + v^T D_{cj}^T Q_k x + x^T Q_k D_{cj} v\big]\mathrm{d}t \bigg\} + \dots \\ & + \sum_{j=1}^{m} \beta_j \bigg\{ \int_{t_{l-1}}^{t_l} \big[x^T(A_{cj}^T Q_l + Q_l A_{cj})x \\ & + v^T D_{cj}^T Q_l x + x^T Q_l D_{cj} v\big]\mathrm{d}t \bigg\} + \dots \\ & + \int_{t_n}^{\infty} \big[x^T(A_{cn}^T Q_n + Q_n A_{cn})x \\ & + v^T D_{cn}^T Q_n x + x^T Q_n D_{cn} v\big]\mathrm{d}t \end{aligned}$$

wherein α_j,$\beta_j > 0$ such that $\sum_{j=1}^n \alpha_j = 1$, and $\sum_{j=1}^n \beta_j = 1$. m and r are the number of neighboring cells to a boundary where the solutions possess infinite switching in finite time (in the time intervals of $[t_{k-1}, t_k]$ and $[t_{l-1}, t_l]$), respectively. With the above formulation, we consider a state evolution scenario including the interior of different cells as well as the facets. Suppose conditions (30) and (31) hold, then it follows that

$$\begin{aligned} & \int_a^b \big[x^T(A_{ci}^T Q_i + Q_i A_{ci})x + v^T D_{ci}^T Q_i x + x^T Q_i D_{ci} v\big]\mathrm{d}t \\ & < \int_a^b \big[x^T(-E_i^T U_i E_i - \eta^{-2} Q_i D_{ci} D_{ci}^T Q_i - C_{ci}^T C_{ci})x \\ & \quad + v^T D_{ci}^T Q_i x + x^T Q_i D_{ci} v + \eta^2 v^T v - \eta^2 v^T v\big]\mathrm{d}t \\ & \le \int_a^b \big[- y^T y + \eta^2 v^T v - \eta^2 (v - \eta^{-2} D_{ci}^T Q_i x)^T \\ & \quad \times (v - \eta^{-2} D_{ci}^T Q_i x)\big]\mathrm{d}t \le \int_a^b (-y^T y + \eta^2 v^T v)\mathrm{d}t \end{aligned}$$

Correspondingly,

$$\begin{aligned} & \sum_{j=1}^n \alpha_j \bigg\{ \int_c^d \big[x^T(A_{cj}^T Q_i + Q_i A_{cj})x \\ & \quad + v^T D_{cj}^T Q_i x + x^T Q_i D_{cj} v\big]\mathrm{d}t \bigg\} \\ & < \sum_{j=1}^n \alpha_j \bigg\{ \int_c^d \big[x^T(-W_{ij} N_{ij}^T - N_{ij} W_{ij}^T - H_{ij}^T T_{ij} H_{ij} \\ & \quad - \eta^{-2} Q_i D_{cj} D_{cj}^T Q_i - C_{cj}^T C_{cj})x + \eta^2 v^T v - \eta^2 v^T v\big]\mathrm{d}t \bigg\} \\ & \le \int_c^d \bigg(- y^T y + \eta^2 v^T v - \sum_{j=1}^n \alpha_j \big(\eta^2 (v - \eta^{-2} D_{cj}^T Q_i x)^T \\ & \quad \times (v - \eta^{-2} D_{cj}^T Q_i x)\big)\bigg)\mathrm{d}t \le \int_c^d (-y^T y + \eta^2 v^T v)\mathrm{d}t \end{aligned}$$

where, $a, b, c, d > 0$ are arbitrary non-negative constants ($b > a$, and $d > c$). Finally, we arrive at the justification that

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}\Gamma}{\mathrm{d}t}\mathrm{d}t \le & \int_0^{t_1} (-y^T y + \eta^2 v^T v)\mathrm{d}t \\ & + \int_{t_1}^{t_2} (-y^T y + \eta^2 v^T v)\mathrm{d}t + \dots \\ & + \int_{t_{k-1}}^{t_k} (-y^T y + \eta^2 v^T v)\mathrm{d}t + \dots \\ & + \int_{t_{l-1}}^{t_l} (-y^T y + \eta^2 v^T v)\mathrm{d}t + \dots \\ & + \int_{t_n}^{\infty} (-y^T y + \eta^2 v^T v)\mathrm{d}t \end{aligned}$$

which reduces to

$$\Gamma\big(x(\infty)\big) - \Gamma\big(x(0)\big) \le \int_0^\infty (-y^T y + \eta^2 v^T v)\mathrm{d}t$$

Moreover, note that $x(\infty) = x(0) = 0$. This can be concluded from the assumption on zero initial conditions, and from the fact that the system is asymptotically stable at origin (as demonstrated earlier in this proof). Consequently, we have

$$0 \le \int_0^\infty (-y^T y + \eta^2 v^T v)\mathrm{d}t$$

which is equivalent to (26). This completes the proof.

Lemma 3. Given a constant $\eta > 0$, the closed loop control system (27) is asymptotically stable at the origin with disturbance attenuation η, if there exist constants $\epsilon_{ij} > 0$, $(i,j) \in \tilde{I}$, $\epsilon_i > 0$, $i \in I$, matrices K_i, $i \in I$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in I$ with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^T S_i E_i > 0 \tag{33}$$

$$\Lambda_i < 0 \tag{34}$$

for all $i \in I$, and

$$\Lambda_{ij} < 0 \tag{35}$$

for all $(i,j) \in \tilde{I}$, where

$$\Lambda_i = \begin{bmatrix} \Pi_i & Q_i & K_i^T B_i^T & K_i^T \mathcal{B}_i^T & K_i^T G_i^T & K_i^T \mathcal{G}_i^T \\ \star & -\Theta_i^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \dfrac{-\epsilon_i}{1+\epsilon_i^2}\mathrm{I} & 0 & 0 & 0 \\ \star & \star & \star & \dfrac{-1}{\epsilon_i}\mathrm{I} & 0 & 0 \\ \star & \star & \star & \star & \dfrac{-\epsilon_i}{2+\epsilon_i+\epsilon_i^2}\mathrm{I} & 0 \\ \star & \star & \star & \star & \star & \dfrac{-\epsilon_i}{1+\epsilon_i+2\epsilon_i^2}\mathrm{I} \end{bmatrix}$$

$$\Lambda_{ij} = \begin{bmatrix} \Pi_{ij} & Q_i & K_j^T B_j^T & K_j^T \mathcal{B}_j^T & K_j^T G_j^T & K_j^T \mathcal{G}_j^T \\ \star & -\Theta_{ij}^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \dfrac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 & 0 \\ \star & \star & \star & \dfrac{-1}{\epsilon_{ij}}\mathrm{I} & 0 & 0 \\ \star & \star & \star & \star & \dfrac{-\epsilon_{ij}}{2+\epsilon_{ij}+\epsilon_{ij}^2}\mathrm{I} & 0 \\ \star & \star & \star & \star & \star & \dfrac{-\epsilon_{ij}}{1+\epsilon_{ij}+2\epsilon_{ij}^2}\mathrm{I} \end{bmatrix}$$

with $\Pi_i = \Xi_i + (1+\frac{3}{\epsilon_i})C_i^T C_i + (1+3\epsilon_i)\mathcal{C}_i^T\mathcal{C}_i$, $\Pi_{ij} = \Xi_{ij} + (1+\frac{3}{\epsilon_{ij}})C_j^T C_j + (1+3\epsilon_{ij})\mathcal{C}_j^T\mathcal{C}_j$, $\Theta_i = (\epsilon_i + \frac{3}{\epsilon_i})I + \eta^{-2}(1+\frac{1}{\epsilon_i})D_i D_i^T + \eta^{-2}(1+\epsilon_i)\mathcal{D}_i\mathcal{D}_i^T$, and $\Theta_{ij} = (\epsilon_{ij} + \frac{3}{\epsilon_{ij}})I + \eta^{-2}(1+\frac{1}{\epsilon_{ij}})D_j D_j^T + \eta^{-2}(1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T$.

Proof. We need to apply Proposition 3. Inequality (33) corresponds to (29). Substituting (28) in (31), the left-hand side of (31) is simplified as $LHS = \big(A_j+\Delta A_j+(B_j+\Delta B_j)K_j\big)^T Q_i + Q_i\big(A_j+\Delta A_j+(B_j+\Delta B_j)K_j\big) + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \eta^{-2}Q_i(D_j+\Delta D_j)(D_j+\Delta D_j)^T Q_i + \big(C_j+\Delta C_j+(G_i+\Delta G_i)K_j\big)^T\big(C_j+\Delta C_j+(G_i+\Delta G_i)K_j\big)$ $\leq A_j^T Q_i + Q_i A_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + K_j^T B_j^T Q_i + Q_i B_j K_j + \frac{2}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij}\mathcal{A}_j^T\mathcal{A}_j + \epsilon_{ij} K_j^T \mathcal{B}_j^T \mathcal{B}_j K_j + \eta^{-2}Q_i\big((1+\frac{1}{\epsilon_{ij}})D_j D_j^T + (1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T\big)Q_i + (1+\epsilon_{ij})C_j^T C_j + (1+\epsilon_{ij})\mathcal{C}_j^T\mathcal{C}_j + \frac{1}{\epsilon_{ij}}C_j^T C_j + \epsilon_{ij}K_j^T G_j^T G_j K_j + \frac{1}{\epsilon_{ij}}C_j^T C_j + \epsilon_{ij}K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j + \epsilon_{ij}\mathcal{C}_j^T\mathcal{C}_j + \frac{1}{\epsilon_{ij}}K_j^T G_j^T G_j K_j + \epsilon_{ij}\mathcal{C}_j^T\mathcal{C}_j + \frac{1}{\epsilon_{ij}}K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j + K_j^T\big((1+\frac{1}{\epsilon_{ij}})G_j^T G_j + (1+\epsilon_{ij})\mathcal{G}_j^T\mathcal{G}_j\big)K_j$.

With some calculation, it can be verified that $LHS \leq \Pi_{ij} + Q_i\big(\frac{2}{\epsilon_{ij}}I + \eta^{-2}(1+\frac{1}{\epsilon_{ij}})D_j^T D_j + \eta^{-2}(1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T\big)Q_i + \epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_j K_j + (\frac{2+\epsilon_{ij}+\epsilon_{ij}^2}{\epsilon_{ij}})K_j^T G_j^T G_j K_j + (\frac{1+\epsilon_{ij}+2\epsilon_{ij}^2}{\epsilon_{ij}})$ $K_j^T\mathcal{G}_j^T\mathcal{G}_j K_j + \frac{1}{\epsilon_{ij}}K_j^T B_j^T B_j K_j + \epsilon_{ij}Q_i Q_i + \frac{1}{\epsilon_{ij}}Q_i Q_i + \epsilon_{ij}K_j^T B_j^T B_j K_j$

which is equivalent to

$$LHS \leq \Pi_{ij} + Q_i\Theta_{ij}Q_i + (\tfrac{1+\epsilon_{ij}^2}{\epsilon_{ij}})K_j^T B_j^T B_j K_j + \epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_j K_j + (\tfrac{2+\epsilon_{ij}+\epsilon_{ij}^2}{\epsilon_{ij}})K_j^T G_j^T G_j K_j + (\tfrac{1+\epsilon_{ij}+2\epsilon_{ij}^2}{\epsilon_{ij}})K_j^T\mathcal{G}_j^T\mathcal{G}_j K_j.$$

Utilizing Shur complement theorem, (35) can be obtained. Thus, if (35) is feasible, then (31) is satisfied. Analogously, it can be proved that (34) is consistent with (30).

It is worth noting that conditions given in Lemma 3 are BMIs in matrix variables K_i and Q_i. In order to deal with the BMI conditions encountered in Lemma 3, the following $V-K$ iteration algorithm is suggested:

- *Initialization*: Select a set of controller gains based on pole placement method or any other controller design scheme to predetermine a set of initial controller gains.
- *Step V*: Given the set of fixed controller gains K_i, $i \in I$, solve the following optimization problem
$$\min_{Q_i,S_i,U_i,T_{ij}} \gamma_i$$
$$\text{subject to } (33), \Lambda_i - \gamma_i\mathrm{I} < 0, \text{and } \Lambda_{ij} - \gamma_i\mathrm{I} < 0$$
for a set of matrices Q_i, $i \in I$.
- *Step K*: Given the set of fixed controller gains Q_i, $i \in I$, solve the following optimization problem
$$\min_{K_i,S_i,U_i,T_{ij}} \gamma_i$$
$$\text{subject to } (33), \Lambda_i - \gamma_i\mathrm{I} < 0, \text{and } \Lambda_{ij} - \gamma_i\mathrm{I} < 0$$
for a set of matrices K_i, $i \in I$.

The algorithm continues till $\gamma_i < 0$, $i \in I$.

5. SIMULATION RESULTS

In this section, we demonstrate the performance of the proposed approach using numerical examples. Example 1 deals with a switched system with Filippov solutions which is asymptotically stable at the origin; but, the disturbance attenuation performance is not satisfactory. Unlike Example 1, Example 2 considers an unstable PWL system in which both asymptotic stability and disturbance mitigation are investigated based on the proposed approach. Not to mention that in both cases uncertainties are also associated with the nominal systems.

5.1 Example 1

Suppose the state-space $X = \mathbb{R}^2$ is divided into four polytopes corresponding to the four quadrants of the second dimensional Euclidean space; i.e,

$$X_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0 \text{ and } x_2 > 0\}$$
$$X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0 \text{ and } x_2 > 0\}$$
$$X_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0 \text{ and } x_2 < 0\}$$
$$X_4 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0 \text{ and } x_2 < 0\} \quad (36)$$

Consider a PWL system with Filippov solutions characterized by (27) and (28) where the associated system matrices are given by

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix}$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}^T$$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^T$$

and the uncertainty bounds specified as

$$\mathcal{A}_1 = \mathcal{A}_3 = \begin{bmatrix} 0 & 0.03 \\ -0.03 & 0 \end{bmatrix}, \mathcal{A}_2 = \mathcal{A}_4 = \begin{bmatrix} 0.03 & 0 \\ 0 & -0.03 \end{bmatrix}$$

$$\mathcal{C}_1 = \mathcal{C}_3 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}^T, \mathcal{C}_2 = \mathcal{C}_4 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}^T$$

The matrices regarding the polytopes can be constructed as

$$E_1 = -E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} E_1 \\ \mathrm{I} \end{bmatrix}, F_2 = \begin{bmatrix} E_2 \\ \mathrm{I} \end{bmatrix}, F_3 = \begin{bmatrix} E_3 \\ \mathrm{I} \end{bmatrix}, F_4 = \begin{bmatrix} E_4 \\ \mathrm{I} \end{bmatrix}$$

$$N_{12} = N_{34} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, N_{14} = N_{23} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H_{12} = -H_{34} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H_{14} = -H_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Based on Lemma 3, a switching controller as defined in (20) is designed in order to ensure that (in addition to preserving the asymptotic stability property of the system) under zero initial conditions the disturbance signal of $v(t) = 5\cos(\pi t)$ is attenuated with $\eta = 0.05$. In this experiment, the constant scalars were preset to $\epsilon_{12} = \epsilon_{23} = \epsilon_{14} = \epsilon_{34} = 1$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 5$. The algorithm was initialized using pole placement method with initial pole positions of $(-1, -2)$ and controller gains of

$$K_1 = K_3 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -3 \\ -5 \end{bmatrix}^T$$

The following solutions was obtained in two iterations

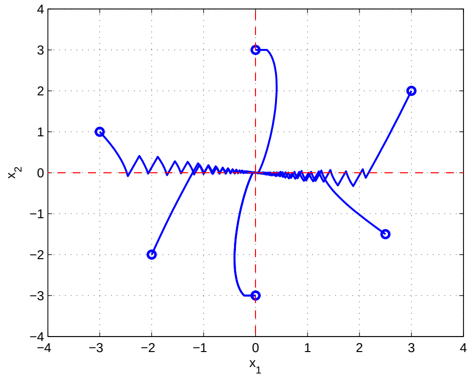

Fig. 1. The trajectories of the closed loop system. The dashed lines illustrate the facets.

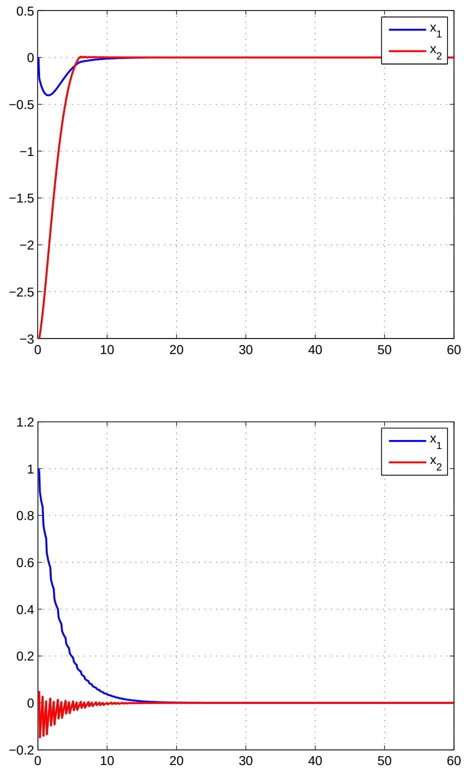

Fig. 2. Evolution of system states when the H_∞ controller is applied: with an initial condition on a non-attractive facet (top) and with an initial condition on an attractive facet (bottom).

$$Q_1 = Q_3 = \begin{bmatrix} 78.29 & 5.96 \\ 5.96 & 3.01 \end{bmatrix} Q_2 = Q_4 = \begin{bmatrix} 33.06 & -1.35 \\ -1.35 & 65.14 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -0.9014 \\ -0.8292 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -0.1137 \\ -0.2715 \end{bmatrix}^T$$

$$\gamma_{min} = -7.03921 \times 10^{-4}$$

Fig. 1. sketches the trajectories of the closed-loop system without disturbance when the H_∞ controller is incorporated. This demonstrates that the Filippov solutions of

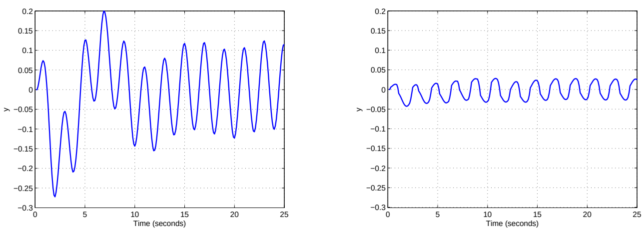

Fig. 3. Response of the closed loop control system with disturbance and zero initial condition: before applying the H_∞ controller (left) and after utilizing the H_∞ controller (right).

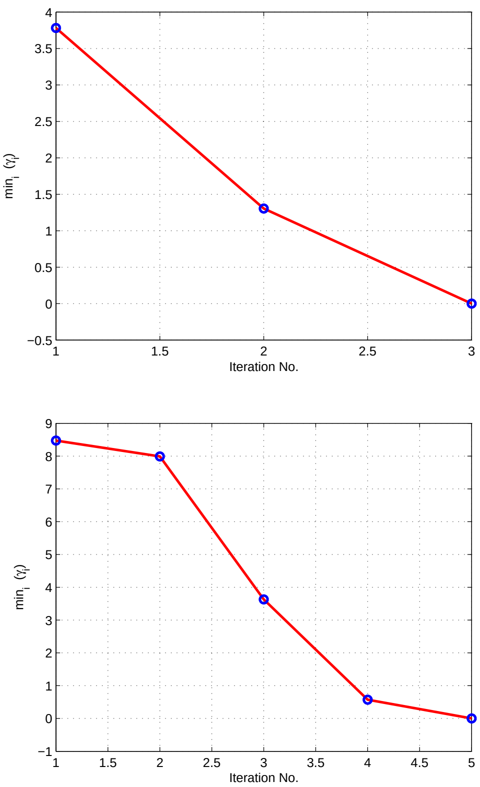

Fig. 4. Convergence performance of the proposed V-K iteration algorithms: the stable controller synthesis (top) and the H_∞ controller synthesis (bottom).

the closed-loop system are asymptotically stable at 0. One should observe that the solutions entering the facet $x_2 = 0$ cannot leave the facet (the so called attractive sliding mode property). This is due to the fact that the velocities at both regions X_1 and X_2 are toward the facet. We emphasize that this result could not been achieved by previous studies which excluded those solutions with infinite switching in finite time. Moreover, Fig. 2. displays the evolution of the states of the closed-loop system.

The disturbance mitigation performance of the proposed method can also be deduced from Fig. 3. As it can be discerned from the figure, the disturbance signal is considerably extenuated as the H_∞ controller is employed.

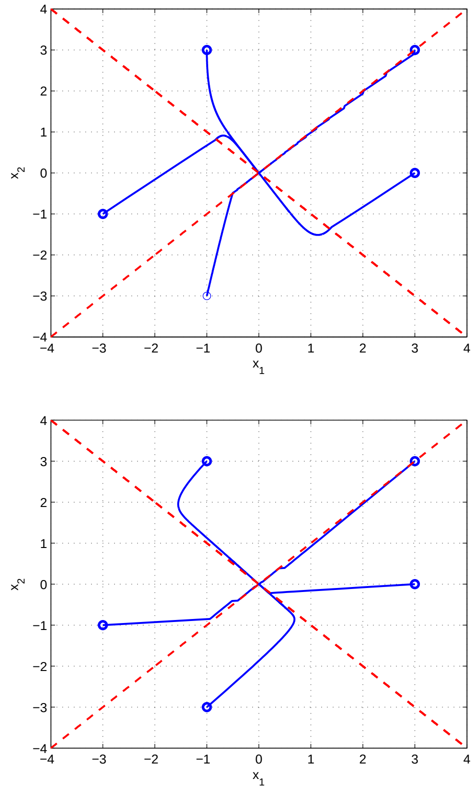

Fig. 5. The trajectories of the closed loop control system: the stable controller synthesis (top) and the H_∞ controller synthesis (bottom). The dashed lines illustrate the facets.

5.2 Example 2

For the sake of comparison, the example used in (Chan et al. (2004)) is selected; but, instead of Carathéodory solutions, Filippov solutions are investigated. Therefore, the system structure has to be modified as delineated next. Consider an uncertain PWL system described by (27) and (28) with $I = \{1, 2, 3, 4\}$ and the state-space is a polyhedral set divided into four polytopes. The associated system matrices are

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

The uncertainty bounds are characterized as

$$\mathcal{A}_1 = \mathcal{A}_3 = \begin{bmatrix} 0 & 0.02 \\ -0.01 & 0 \end{bmatrix}, \mathcal{A}_2 = \mathcal{A}_4 = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.02 \end{bmatrix}$$

$$\mathcal{B}_1 = \mathcal{B}_3 = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \mathcal{B}_2 = \mathcal{B}_4 = \begin{bmatrix} 0.02 \\ 0 \end{bmatrix}$$

The matrices characterizing the polytopes are given as follows

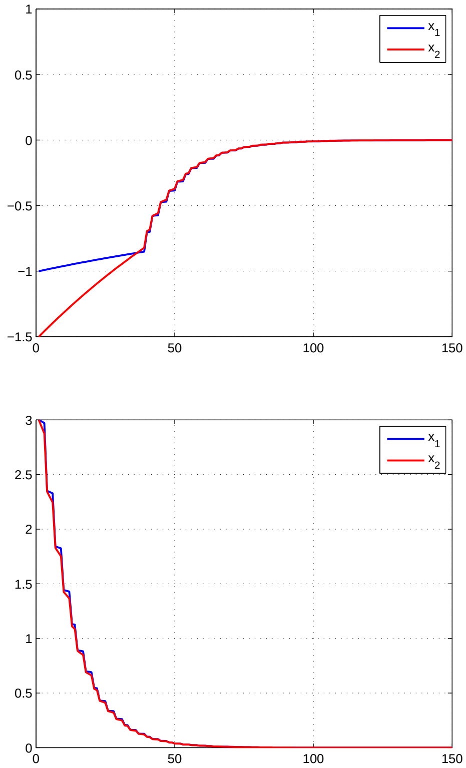

Fig. 6. Evolution of system states when the H_∞ controller is applied: with an initial condition in the interior of a polytope (top) and with an initial condition on an attractive facet (bottom).

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} E_1 \\ \mathrm{I} \end{bmatrix}, F_2 = \begin{bmatrix} E_2 \\ \mathrm{I} \end{bmatrix}, F_3 = \begin{bmatrix} E_3 \\ \mathrm{I} \end{bmatrix}, F_4 = \begin{bmatrix} E_4 \\ \mathrm{I} \end{bmatrix}$$

$$N_{12} = N_{34} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, N_{14} = N_{23} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$H_{12} = -H_{34} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, H_{14} = -H_{23} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

It is worth noting that the open-loop system is unstable and since solutions with infinite switching in finite time are present the approach reported in (Chan et al. (2004)) and common Lyapunov based methods are not applicable. The $V - K$ iteration algorithm is initialized using pole placement method. The assigned closed-loop poles for the dynamics in each polytope are $(-3, -2)$ and the corresponding initial controller gains are

$$K_1 = K_3 = \begin{bmatrix} -119.5 \\ -7 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -5 \\ 19.5 \end{bmatrix}^T$$

Using the scheme presented in this paper for a set of constants $\epsilon_{12} = \epsilon_{23} = \epsilon_{14} = \epsilon_{34} = 10$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 100$, the following solutions has been obtained

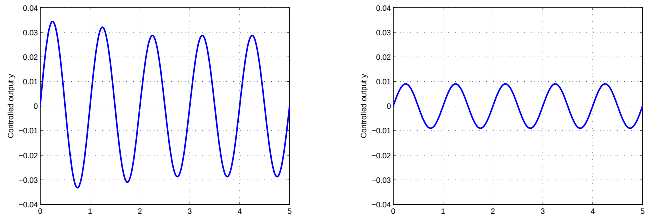

Fig. 7. Response of the closed loop control system with disturbance and zero initial condition: the stable controller synthesis (left) and the H_∞ controller synthesis (right).

$$Q_1 = Q_3 = \begin{bmatrix} 135.26 & 2.18 \\ 2.18 & 1.83 \end{bmatrix}, Q_2 = Q_4 = \begin{bmatrix} 84.67 & -5.43 \\ -5.43 & 707.09 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -389.92 \\ -30.14 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -12.88 \\ -0.56 \end{bmatrix}^T$$

$$\gamma_{min} = -4.5328 \times 10^{-4}$$

for the stable controller synthesis in three iterations and

$$Q_1 = Q_3 = \begin{bmatrix} 463.75 & 24.94 \\ 24.94 & 2.39 \end{bmatrix}, Q_2 = Q_4 = \begin{bmatrix} 52.26 & -7.39 \\ -7.39 & 763.47 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -637.72 \\ -30.14 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -21.53 \\ -1.69 \end{bmatrix}^T$$

$$\gamma_{min} = -2.7186 \times 10^{-5}$$

for the H_∞ controller design with $\eta = 0.1$ in five iterations. Consequently, it follows from Lemma 3 that the closed loop control system is asymptotically stable at the origin and the disturbance attenuation criterion is satisfied. The convergence performance of the proposed V-K iteration algorithms are depicted in Fig. 4. The V-K algorithm pertaining to stabilizing controller and H_∞ controller took 17.5947 seconds and 31.8351 seconds respectively on Intel(R) Core(TM)2 Due CPU T7500 @2.20GHz and 3.00 GB of RAM using MATLAB R2010b. Fig. 5. portrays the simulation results of four different initial conditions (in the absence of disturbance) which prove the stability of the closed loop systems. Notice, in particular, that solutions with infinite switching in finite time on facets are also present (see Fig. 6.). This should be opposed to the results in (Chan et al. (2004)) where only Carathódory solutions are taken into account. Additionally, the simulation results in the presence of disturbance ($v(t) = 4\sin(2\pi t)$) and zero initial conditions are illustrated in Fig. 7. which ascertains the disturbance attenuation performance of the proposed controller.

6. CONCLUSIONS

In this paper, the stability and control problem of PWL and uncertain PWL systems with Filippov Solutions was considered. The foremost purpose of this research was to extend the previous results on PWL systems to the case of solutions with infinite switching in finite time and sliding motions. In this regard, we have proposed a set of matrix inequality conditions to investigate the stability of a PWL or uncertain PWL system. Additionally, two methods based on BMIs are devised for the synthesis of stable and robust H_∞ controllers for PWL and uncertain PWL systems with Filippov solutions. These schemes has been examined through simulation experiments.

REFERENCES

Antwerp, J.V. and Braatz, R. (2000). A toturial on linear and bilinear matrix inequalities. *J. Process Control*, 10, 363–385.

Aubin, J. and Cellina, A. (1984). *Differential Inclusions, Set-Valued Maps And Viability Theory*, volume 264 of *Grundl. der Math. Wiss.* Springer-Verlag, Berlin.

Azuma, S., Yanagisawa, E., and Imura, J. (2008). Controllability analysis of biosystems based on piecewise affine systems approach. *IEEE Trans. Automatic Control, Special Issue on Systems Biology*, 139–159.

Boiko, I. (2009). *Discontinuous Control Systems.* Birkhäuser, Boston.

Branicky, M. (1998). Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. on Automatic Control*, 43(4), 475–482.

Chan, M., Zhu, C., and Feng, G. (2004). Linear-matrix-inequality-based approach to H_∞ controller synthesis of uncertain continuous-time piecewise linear systems. *IEE Proc.- Control Theory App.*, 151(3), 295–301.

Cortes, J. (1998). Discontinuous dynamical systems: A tutorial on solutions, non-smooth analysis, and stability. *IEEE Control Syst. Mag.*, 28(3), 36–73.

de Best, J., Bukkems, B., van de Molengraft, M., and Heemels, W. (2008). Robust control of piecewise linear systems: A case study in sheet flow control. *Control Engineering Practice*, 16, 991–1003.

Dobrescu, R., Oltean, V., and Dobrescu, M. (2008). Simultation models and zeno path avoidance in a class of piecewise linear biochemical process. *J. of Control Engineering and Applied Informatics*, 10(1), 33–39.

Dullerud, G. and Paganini, F. (2000). *A Course in Robust Control Theory.* Springer.

Edwards, C., Colet, E., and Fridman, L. (eds.) (2006). *Advances in Variable Structure and Sliding Mode Control.* Lecture Notes in Control and Information Science. Springer, Berlin.

Filippov, A. (1988). *Differential equations with discontinuous right-hand sides*, volume 18 of *Mathematics and its applications (Soviet Series).* Kluwer Academic Publishers Group, Dordecht.

Forti, M. and Nistri, P. (2003). Global convergence of neural networks with discontinuous neuron activations. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 50(11), 1421–1435.

Giaouris, D., Banerjee, S., Zahawi, B., and Pickert, V. (2008). Stability analysis of the continuous-conduction-mode buck converter via Filippov's method. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 55(4), 1084–1096.

Goncalves, J., Megretski, A., and Dahleh, M.A. (2003). Global analysis of piecewise linear systems using impact maps and surface lyapunov functions. *IEEE Trans. Automatic Control*, 48(12), 2089–2106.

Hassibi, A. and Boyd, S. (1998). Quadratic stabilization and control of piecewise-linear systems. In *American Control Conference*, volume 6, 3659–3664.

Jing, H., Liu, Z., and Chen, H. (2011). A switched control strategy for antilock braking system with on/off valves. *IEEE Transactions on Vehicular Technology*, 60(4), 1470–1484.

Johansson, M. (2003). *Piecewise Linear Control Systems.* Springer-Verlag, Berlin.

Johansson, M. and Rantzer, A. (1998). Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Trans. Automatic Control*, 43(2), 555–559.

Leth, J. and Wisniewski, R. (2012). On formalism and stability of switched systems. *J. Contol Theory and Applications*, 10(2).

Ohta, Y. and Yokoyama, H. (2010). Stability analysis of uncertain piecewise linear systems using piecewise quadratic lyapunov functions. In *IEEE International Symposium on Intelligent Control*, 2112–2117.

Paden, B. and Sastry, S. (1987). A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, 34(1), 73–82.

Polik, I. and Terlaky, T. (2007). A survey of the s-lemma. *SIAM Review*, 49(3), 371–418.

Serbanescu, A. and Popeea, C. (2004). Specific H_∞ synthesis for time delay systems. *J. of Control Engineering and Applied Informatics*, 6(1), 39–52.

Stoica, A. (2005). State feedback Q-stabilization with robust H_∞ performance for systems with LFT based parametric uncertainty. *J. of Control Engineering and Applied Informatics*, 7(2), 3–9.

Sun, Z. (2010). Stability of piecewise linear systems revisited. *Annual Reviews in Control*, 34, 221–231.

Yakubovich, V., Leonov, G., and Gelig, A. (2004). *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, volume 14 of *Stability, Vibration and Control of Systems.* World Scientific Publishing Co., Singapore.