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Robust H_∞ control of uncertain switched systems defined on polyhedral sets with Filippov solutions

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ABSTRACT

This paper considers the control problem of a class of uncertain switched systems defined on polyhedral sets known as piecewise linear systems where, instead of the conventional Carathéodory solutions, Filippov solutions are studied. In other words, in contrast to the previous studies, solutions with infinite switching in finite time along the facets and on faces of arbitrary dimensions are also taken into account. Firstly, established upon previous studies, a set of linear matrix inequalities are brought forward which determines the asymptotic stability of piecewise linear systems with Filippov solutions. Subsequently, bilinear matrix inequality conditions for synthesizing a robust controller with a guaranteed H_∞ performance are presented. Furthermore, these results has been generalized to the case of piecewise affine systems. Finally, a V–K iteration algorithm is proposed to deal with the aforementioned bilinear matrix inequalities. The validity of the proposed method is verified through the analysis of two simulation examples.

1. Introduction

Piecewise linear (PWL) systems are an important class of hybrid systems, which have received tremendous attention in open literature [1–13]. By a PWL system, we understand a family of linear systems defined on polyhedral sets such that the dynamics inside a polytope is governed by a linear dynamic equation. The union of these polyhedral sets forms the state-space. We say that a “switch” has occurred whenever a trajectory passes to an adjacent polytope.

The stability analysis of PWL systems is an intricate assignment. It is established that even if all the subsystems are stable, the overall system may possess divergent trajectories [11]. Furthermore, the behavior of solutions along the facets may engender unstable trajectories where transitions are, generally speaking, multi-valued. That is, a PWL system with stable Carathéodory solutions may possess divergent Filippov solutions such that the overall system is unstable (see Example 5 in [8]). Hence, the stability of Carathédory solutions does not imply the stability of the overall PWL system.

The stability problem of PWL systems has been addressed by a number of researchers. An efficacious contribution was made by Johansson and Rantzer [4]. The authors proposed a number of linear matrix inequality (LMI) feasibility tests to investigate the exponential stability of a given PWL system by introducing the concept of piecewise quadratic Lyapunov functions. Following the same trend, [6] extended the results to the case of uncertain PWL systems. The authors also brought forward an H_∞ controller synthesis scheme for uncertain PWL systems based on a set of LMI conditions. In [14], the stability issue of uncertain PWL systems with time-delay has been treated. The ultimate boundedness property of large-scale arrays consisting of piecewise affine subsystems linearly interconnected through channels with delays has also been investigated in [15].

However, the solutions considered implicitly in the mentioned contributions are defined in the sense of Carathéodory. This means that a solution of a PWL system is the concatenation of classical solutions on the facets of polyhedral sets. In other words, sliding phenomena or solutions with infinite switching in finite time are inevitably eliminated from the analyses. In this study, in lieu of the Carathéodory solutions, the more universal Filippov solutions [16] are considered and analyzed. This is motivated by recent trends in discontinuous control systems [17] and the renowned sliding mode control techniques [18]. Our approach has its roots in the results reported by [8], wherein the authors applied the theory of differential inclusions to derive stability theorems for switched systems with Filippov solutions. In this regard, we propose a methodology to synthesize robust controllers with H_∞ performance. The results reported in this paper are formulated as a set of LMI or bilinear matrix inequality (BMI) conditions which can be formulated into a semi-definite programming problem. It is also shown that with slight modifications

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the same results can be utilized to analyze piecewise affine (PWA) systems.

The framework of this paper is organized as follows. A brief introduction to polyhedral sets and the notations used in this paper are presented in the subsequent section. The stability problem of PWL systems is addressed in Section 3. The H_∞ Controller synthesis methodology and a V–K iteration algorithm to deal with the BMI conditions are described in Section 4. The accuracy of the proposed method is evaluated by two simulation examples in Section 5. The paper ends with conclusions in Section 6.

2. Notations and definitions

A polyhedral set is defined by finitely many linear inequalities $\{x \in \mathbb{R}^n | Ex \succcurlyeq e\}$ with $E \in \mathbb{R}^{s \times n}$ and $e \in \mathbb{R}^s$ where the notation $\succcurlyeq$ signifies the component-wise inequality. This definition connotes that a polyhedra is the intersection of a finite number of half-spaces. A polytope is a bounded polyhedral set or equivalently the convex hull of finitely many points. Suppose X be a polyhedral set and assume $\mathcal{H}$ be a halfspace such that $X \subset \mathcal{H}$. Let $X^F = X \cap \mathcal{H}$ be non-empty. Then, the polyhedron X^F is called a (proper) face of X. Obviously, the improper faces of X are the subsets $\emptyset$ and X. Faces of dimension $dim(X)-1$ are called facets [19].

In this study, we will consider a class of switched systems with Filippov solutions $\mathcal{S} = \{\mathcal{X},\mathcal{U},\mathcal{V},X,I,F,G\}$, where $\mathcal{X} \subset \mathbb{R}^n$ is a polyhedral set representing the state space, $\mathcal{X} = \{X_i\}_{i \in I}$ is the set containing the polytopes in $\mathcal{X}$ with index set $I = \{1, 2, \ldots, n_X\}$ (note that $\bigcup_{i \in I} X_i = \mathcal{X}$). Each polytope X_i is characterized by the set $\{x \in \mathcal{X} | E_i x \succcurlyeq 0\}$. $\mathcal{U}$ is the control space and $\mathcal{V}$ is the disturbance space, which are both subsets of Euclidean spaces. In addition, each function $v(t)$ belongs to the class of square integrable functions $L_2[0,\infty)$; i.e., the class of functions for which

$$\|v\|_{L_2} = \left(\int_0^\infty v(t)^T v(t) \, \mathrm{d}t \right)^{1/2}$$

is well-defined and finite. $F = \{f_i\}_{i \in I}$ and $G = \{g_i\}_{i \in I}$ are families of linear functions associated with the system states x and outputs y. Each f_i consists of six elements $(A_i,B_i,D_i; \Delta A_i,\Delta B_i,\Delta D_i)$ and each g_i is composed of four elements $(C_i,G_i; \Delta C_i,\Delta G_i)$. Furthermore, $f_i : Y_i \times \mathcal{U} \times \mathcal{V} \to \mathbb{R}^n; (x,u,v) \mapsto \{z \in \mathbb{R}^n | z = (A_i+\Delta A_i)x + (B_i+\Delta B_i)u + (D_i+\Delta D_i)v\}$ and $g_i : Y_i \times \mathcal{U} \to \mathbb{R}^m; (x,u) \mapsto \{z \in \mathbb{R}^m | z = (C_i+\Delta C_i)x + (G_i+\Delta G_i)u\}$ where Y_i is an open neighborhood of X_i. The set of matrices (A_i,B_i,C_i,D_i,G_i) are defined over the polytope X_i and $(\Delta A_i,\Delta B_i, \Delta C_i, \Delta D_i,\Delta G_i)$ encompass the corresponding uncertainty terms. In order to derive the stability and control results, we assume that the upper bound of uncertainties are known apriori; i.e.,

$$\Delta A_i^T \Delta A_i \le \mathcal{A}_i^T \mathcal{A}_i$$

$$\Delta B_i^T \Delta B_i \le \mathcal{B}_i^T \mathcal{B}_i$$

$$\Delta C_i^T \Delta C_i \le \mathcal{C}_i^T \mathcal{C}_i$$

$$\Delta D_i^T \Delta D_i \le \mathcal{D}_i^T \mathcal{D}_i$$

$$\Delta G_i^T \Delta G_i \le \mathcal{G}_i^T \mathcal{G}_i \tag{1}$$

in which $(\mathcal{A}_i,\mathcal{B}_i,\mathcal{C}_i,\mathcal{D}_i,\mathcal{G}_i)$ are any set of constant matrices with the same dimension as (A_i,B_i,C_i,D_i,G_i) satisfying (1).

The dynamics of the system can be described by

$$\dot{x}(t) \in co(\mathcal{F}(x(t),u(t),v(t))) \tag{2}$$

$$y(t) \in \mathcal{G}(x(t),u(t)) \tag{3}$$

where $co(\cdot)$ denotes the convex hull, the set valued maps [20] $\mathcal{F}$ and $\mathcal{G}$ are defined as

$$\mathcal{F} : \mathcal{X} \times \mathcal{U} \times \mathcal{V} \to 2^{\mathcal{X}}; (x,u,v) \mapsto \{z \in \mathbb{R}^n | z = f_i(x,u,v) \text{ if } x \in X_i\} \tag{4}$$

$$\mathcal{G} : \mathcal{X} \times \mathcal{U} \to 2^{\mathbb{R}^m}; (x,u) \mapsto \{z \in \mathbb{R}^m | z = g_i(x,u) \text{ if } x \in X_i\} \tag{5}$$

where the notation 2^A means the power set or the set of all subsets of A. Denote by $\tilde{I} = \{(i,j) \in I^2 | X_i \cap X_j \ne \emptyset, i \ne j\}$ the set of index pairs which determines the polytopes with non-empty intersections. We now assume that each polytope is the intersection of a finite set of supporting halfspaces. By N_{ij} denote the normal vector pertained to the hyperplane supporting both X_i and X_j. Consequently, each boundary can be characterized as

$$X_i \cap X_j = \{x \in \mathcal{X} | N_{ij}^T x \approx 0,\ H_{ij} x \succcurlyeq 0,\ (i,j) \in \tilde{I}\} \tag{6}$$

where $\approx$ represent the component-wise equality and the inequality $H_{ij}x \succcurlyeq 0$ confines the hyperplane to the interested region. Throughout the paper, the matrix inequalities should be understood in the sense of positive definiteness; i.e., $A > B$ ($A \ge B$) means $A-B$ is positive definite (semi-positive definite). In case of matrix inequalities, I denotes the unity matrix (the size of I can be inferred from the context) and should be distinguished from the index set I. In matrices, $\star$ in place of a matrix entry a_{mn} means that $a_{mn} = a_{nm}^T$.

A Filippov solution to (2) is an absolutely continuous function $[0,T) \to \mathcal{X}; t \mapsto \phi(t)$ $(T > 0)$ which solves the following Cauchy problem

$$\dot{\phi}(t) \in co(\mathcal{F}(\phi(t),u(t),v(t))) \quad \text{a.e.,} \quad \phi(0) = \phi_0 \tag{7}$$

In the sequel, it is assumed that at any interior point $x \in \mathcal{X}$ there exists a Filippov solution to system (1). This can be evidenced by Proposition 5 in [8]. For more information pertaining to the solutions and their existence or uniqueness properties, the interested reader is referred to the expository review [21] and the didactic book [16].

3. Stability of PWL systems with Filippov Solutions

In [8], a stability theorem for switched systems defined on polyhedral sets in the context of Filippov solutions is proposed. In what follows, we reformulate this latter stability theorem in terms of matrix inequalities which provides computationally doable means to inspect the robust stability of uncertain switched systems. These matrix inequalities would be later utilized to devise a stabilizing controller with H_∞ disturbance rejection performance.

Lemma 1. *Consider the following autonomous PWL system*

$$\dot{x} \in co(\mathcal{F}(x)) \tag{8}$$

with $\Delta A_i \approx 0$. If there exists quadratic forms $\Phi_i(x) = x^T Q_i x$, $\Psi_i(x) = x^T(A_i^T Q_i + Q_i A_i)x$ and $\Psi_{ij}(x) = x^T(A_j^T Q_i + Q_i A_j)x$ satisfying

$$\Phi_i(x) > 0 \quad \text{for all} \quad x \in X_i \backslash \{\mathbf{0}\} \tag{9}$$

$$\Psi_i(x) < 0 \quad \text{for all} \quad x \in X_i \backslash \{\mathbf{0}\} \tag{10}$$

for all $i \in I$, and

$$\Psi_{ij}(x) < 0 \quad \text{for all} \quad x \in X_i \cap X_j \backslash \{\mathbf{0}\} \tag{11}$$

$$\Phi_i(x) = \Phi_j(x) \quad \text{for all} \quad x \in X_i \cap X_j \tag{12}$$

for all $(i,j) \in \tilde{I}$. Then, the equilibrium point $\mathbf{0}$ of (8) is asymptotically stable.

Remark 1. The inclusions $x \in X_i \backslash \{\mathbf{0}\}$ and $x \in X_i \cap X_j$ are analogous to $\{x \in \mathcal{X} | E_i x \succ 0\}$ and (6), respectively.

It is worth noting that Conditions (9)–(10) are concerned with the positivity of a quadratic form over a polytope; whereas, (11) is about positivity over a hyperplane. Condition (12) asserts that the candidate Lyapunov functions should be continuous (along the facets). A well known LMI formulation of conditions (9), (10) and (12) was proposed in [4] which is described next. Let us construct a set of matrices F_i, $i \in I$ such that $F_i x = F_j x$ for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$. Then, it follows that the piecewise linear candidate Lyapunov functions can be formulated as

$$V(x) = x^T F_i^T M F_i x = x^T Q_i x \quad \text{if} \quad x \in X_i \tag{13}$$

where the free parameters of Lyapunov functions are concentrated in the symmetric matrix M. In the following lemma we generalize the results proposed by [4] to PWL systems with the more general Filippov solutions.

Lemma 2. *Consider the PWL system* (8) *with Fillipov solutions, and the family of piecewise quadratic Lyapunov functions* $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$. *If there exist a set of symmetric matrices* Q_i, *three sets of symmetric matrices* U_i, S_i, T_{ij} *with non-negative entries, and matrices* W_{ij} *of appropriate dimensions with* $i \in I$ *and* $(i,j) \in \tilde{I}$, *such that the following LMI problem is feasible*

$$Q_i - E_i^T S_i E_i > 0 \tag{14}$$

$$A_i^T Q_i + Q_i A_i + E_i^T U_i E_i < 0 \tag{15}$$

for all $i \in I$, *and*

$$A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} < 0 \tag{16}$$

for all $(i,j) \in \tilde{I}$. *Then, the equilibrium point* $\mathbf{0}$ *of* (8) *is asymptotically stable.*

Proof. Matrix inequalities (14) and (15) are the same as Eq. (11) in Theorem 1 in [4] which satisfy (9)–(10). The continuity of the Lyapunov functions is also ensured from the assumption that $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$ since $F_i x = F_j x$, for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$. (11) is equivalent to $x^T (A_j^T Q_i + Q_i A_j) x < 0$ for $\{x \in \mathcal{X} | N_{ij}^T x \approx 0, H_{ij} x \succ 0\}$. Applying the S-procedure and Finsler's lemma [22], we obtain (16) for a set of matrices T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries and W_{ij}, $(i,j) \in \tilde{I}$ with appropriate dimensions. □

We remark that algorithms for constructing matrices E_i and F_i, $i \in I$, are described in [9].

Remark 2. A similar LMI formulation to (11) can be found in [9]; whereas, our analysis, in this paper, is established upon the stability theorem delineated in Proposition 10 in [8] which considered the Filippov Solutions.

4. Robust controller synthesis with H_∞ performance

In this section, we propose a set of conditions to design a robust stabilizing switching controller of the form

$$u \in \mathcal{K}(x)$$

$$\mathcal{K} : \mathcal{X} \to 2^{\mathcal{U}}; x \mapsto \{z \in \mathcal{U} | z = K_i x \text{ if } x \in X_i\} \tag{17}$$

with a guaranteed H_∞ performance [23]. That is, a controller such that, in addition to asymptotic stability ($\lim_{t \to \infty} \phi(t) = \mathbf{0}$ for all $\Phi(t)$ satisfying (7)), ensures that the induced L_2-norm of the operator from $\nu(t)$ to the controller output $y(t)$ is less than a constant $\eta > 0$ under zero initial conditions ($x(0) = \mathbf{0}$); in other words,

$$\|y\|_{L_2} \le \eta \|\nu\|_{L_2} \tag{18}$$

given any non-zero $\nu \in L_2[0,\infty)$.

If we apply the switching controller (17) to (2) and (3), we arrive at the following controlled system with outputs

$$\dot{x}(t) \in co(\tilde{\mathcal{F}}(x(t), \nu(t)))$$

$$y(t) \in \tilde{\mathcal{G}}(x(t)) \tag{19}$$

where $\tilde{\mathcal{F}} : \mathcal{X} \times \mathcal{V} \to 2^{\mathcal{X}}; (x,\nu) \mapsto \{z \in \mathbb{R}^n | z = A_{ci} x + D_{ci} \nu \text{ if } x \in X_i\}$ and $\mathcal{G} : \mathcal{X} \to 2^{\mathbb{R}^m}; x \mapsto \{z \in \mathbb{R}^m | z = C_{ci}(x) \text{ if } x \in X_i\}$ with

$$A_{ci} = A_i + \Delta A_i + (B_i + \Delta B_i) K_i$$

$$D_{ci} = D_i + \Delta D_i$$

$$C_{ci} = C_i + \Delta C_i + (G_i + \Delta G_i) K_i \tag{20}$$

Theorem 1. *System* (19) *is asymptotically stable at the origin with disturbance attenuation* η *as defined in* (18), *if there exist a set of symmetric matrices* Q_i, $i \in I$, *three sets of symmetric matrices* U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$ *with non-negative entries, and matrices* W_{ij}, $(i,j) \in \tilde{I}$ *of appropriate dimensions such that*

$$Q_i - E_i^T S_i E_i > 0 \tag{21}$$

$$A_{ci}^T Q_i + Q_i A_{ci} + E_i^T U_i E_i + \eta^{-2} Q_i D_{ci} D_{ci}^T Q_i + C_{ci}^T C_{ci} < 0 \tag{22}$$

for all $i \in I$, *and*

$$A_{cj}^T Q_i + Q_i A_{cj} + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \eta^{-2} Q_i D_{cj} D_{cj}^T Q_i + C_{cj}^T C_{cj} < 0 \tag{23}$$

for all $(i,j) \in \tilde{I}$.

Proof. Refer to Appendix A.

Theorem 2. *Given a constant* $\eta > 0$, *the closed loop control system* (19) *is asymptotically stable at the origin with disturbance attenuation* η, *if there exist constants* $\epsilon_{ij} > 0$, $(i,j) \in \tilde{I}$, $\epsilon_i > 0$, $i \in I$, *matrices* K_i, $i \in I$, *a set of symmetric matrices* Q_i, $i \in I$, *three sets of symmetric matrices* U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in I$ *with non-negative entries, and matrices* W_{ij}, $(i,j) \in \tilde{I}$ *of appropriate dimensions such that*

$$Q_i - E_i^T S_i E_i > 0 \tag{24}$$

$$\Lambda_i < 0 \tag{25}$$

for all $i \in I$, *and*

$$\Lambda_{ij} < 0 \tag{26}$$

for all $(i,j) \in \tilde{I}$, *where*

$$\Lambda_i = \begin{bmatrix} \Pi_i & Q_i & K_i^T B_i^T & K_i^T \mathcal{B}_i^T & K_i^T G_i^T & K_i^T \mathcal{G}_i^T \\ \star & -\Theta_i^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \dfrac{-\epsilon_i}{1+\epsilon_i^2} \mathrm{I} & 0 & 0 & 0 \\ \star & \star & \star & \dfrac{-1}{\epsilon_i} \mathrm{I} & 0 & 0 \\ \star & \star & \star & \star & \dfrac{-\epsilon_i}{2+\epsilon_i+\epsilon_i^2} \mathrm{I} & 0 \\ \star & \star & \star & \star & \star & \dfrac{-\epsilon_i}{1+\epsilon_i+2\epsilon_i^2} \mathrm{I} \end{bmatrix}$$

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$$\Lambda_{ij}=\begin{bmatrix}\Pi_{ij} & Q_i & K_j^TB_j^T & K_j^T\mathcal{B}_j^T & K_j^TG_j^T & K_j^T\mathcal{G}_j^T\\ \star & -\Theta_{ij}^{-1} & 0 & 0 & 0 & 0\\ \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 & 0\\ \star & \star & \star & \frac{-1}{\epsilon_{ij}}\mathrm{I} & 0 & 0\\ \star & \star & \star & \star & \frac{-\epsilon_{ij}}{2+\epsilon_{ij}+\epsilon_{ij}^2}\mathrm{I} & 0\\ \star & \star & \star & \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}+2\epsilon_{ij}^2}\mathrm{I}\end{bmatrix}$$

with

$$\Pi_i=A_i^TQ_i+Q_iA_i+E_i^TU_iE_i+\epsilon_i\mathcal{A}_i^T\mathcal{A}_i+\left(1+\frac{3}{\epsilon_i}\right)C_i^TC_i+(1+3\epsilon_i)\mathcal{C}_i^T\mathcal{C}_i$$

$$\Pi_{ij}=A_j^TQ_i+Q_iA_j+W_{ij}N_{ij}^T+N_{ij}W_{ij}^T+H_{ij}^TT_{ij}H_{ij}+\epsilon_{ij}\mathcal{A}_j^T\mathcal{A}_j+\left(1+\frac{3}{\epsilon_{ij}}\right)C_j^TC_j+(1+3\epsilon_{ij})\mathcal{C}_j^T\mathcal{C}_j$$

$$\Theta_i=\left(\epsilon_i+\frac{3}{\epsilon_i}\right)I+\eta^{-2}\left(1+\frac{1}{\epsilon_i}\right)D_iD_i^T+\eta^{-2}(1+\epsilon_i)\mathcal{D}_i\mathcal{D}_i^T$$

and

$$\Theta_{ij}=\left(\epsilon_{ij}+\frac{3}{\epsilon_{ij}}\right)I+\eta^{-2}\left(1+\frac{1}{\epsilon_{ij}}\right)D_jD_j^T+\eta^{-2}(1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T$$

Proof. We need to apply Theorem 1. Inequality (24) corresponds to (21). Substituting (27) in (23), the left-hand side of (23) is simplified as $LHS=(A_j+\Delta A_j+(B_j+\Delta B_j)K_j)^TQ_i+Q_i(A_j+\Delta A_j+(B_j+\Delta B_j)K_j)+W_{ij}N_{ij}^T+N_{ij}W_{ij}^T+H_{ij}^TT_{ij}H_{ij}+\eta^{-2}Q_i(D_j+\Delta D_j)(D_j+\Delta D_j)^TQ_i+(C_j+\Delta C_j+(G_i+\Delta G_i)K_j)^T(C_j+\Delta C_j+(G_i+\Delta G_i)K_j)\le A_j^TQ_i+Q_iA_j+W_{ij}N_{ij}^T+N_{ij}W_{ij}^T+H_{ij}^TT_{ij}H_{ij}+K_j^TB_j^TQ_i+Q_iB_jK_j+2/\epsilon_{ij}Q_iQ_i+\epsilon_{ij}\mathcal{A}_j^T\mathcal{A}_j+\epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_jK_j+\eta^{-2}Q_i((1+1/\epsilon_{ij})D_jD_j^T+(1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T)Q_i+(1+\epsilon_{ij})C_j^TC_j+(1+\epsilon_{ij})\mathcal{C}_j^T\mathcal{C}_j+1/\epsilon_{ij}C_j^TC_j+\epsilon_{ij}K_j^TG_j^TG_jK_j+1/\epsilon_{ij}C_j^TC_j+\epsilon_{ij}K_j^T\mathcal{G}_j^T\mathcal{G}_jK_j+\epsilon_{ij}\mathcal{C}_j^T\mathcal{C}_j+1/\epsilon_{ij}K_j^TG_j^TG_jK_j+\epsilon_{ij}\mathcal{C}_j^T\mathcal{C}_j+1/\epsilon_{ij}K_j^T\mathcal{G}_j^T\mathcal{G}_jK_j+K_j^T((1+1/\epsilon_{ij})G_j^TG_j+(1+\epsilon_{ij})\mathcal{G}_j^T\mathcal{G}_j)K_j$.

With some calculation, it can be verified that

$$LHS\le\Pi_{ij}+Q_i\left(\frac{2}{\epsilon_{ij}}I+\eta^{-2}\left(1+\frac{1}{\epsilon_{ij}}\right)D_j^TD_j+\eta^{-2}(1+\epsilon_{ij})\mathcal{D}_j\mathcal{D}_j^T\right)Q_i+\epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_jK_j+\left(\frac{2+\epsilon_{ij}+\epsilon_{ij}^2}{\epsilon_{ij}}\right)K_j^TG_j^TG_jK_j+\left(\frac{1+\epsilon_{ij}+2\epsilon_{ij}^2}{\epsilon_{ij}}\right)K_j^T\mathcal{G}_j^T\mathcal{G}_jK_j+\frac{1}{\epsilon_{ij}}K_j^TB_j^TB_jK_j+\epsilon_{ij}Q_iQ_i+\frac{1}{\epsilon_{ij}}Q_iQ_i+\epsilon_{ij}K_j^TB_j^TB_jK_j$$

which is equivalent to $LHS\le\Pi_{ij}+Q_i\Theta_{ij}Q_i+((1+\epsilon_{ij}^2)/\epsilon_{ij})K_j^TB_j^TB_jK_j+\epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_jK_j+((2+\epsilon_{ij}+\epsilon_{ij}^2)/\epsilon_{ij})K_j^TG_j^TG_jK_j+((1+\epsilon_{ij}+2\epsilon_{ij}^2)/\epsilon_{ij})K_j^T\mathcal{G}_j^T\mathcal{G}_jK_j$.

Utilizing Shur complement theorem, (26) can be obtained. Thus, if (26) is feasible, then (23) is satisfied. Analogously, it can be proved that (25) is consistent with (22). □

Remark 3. The conditions derived in Theorem 2 are BMIs [24] in the variables Q_i and K_i.

In order to deal with the BMI conditions encountered in Theorem 2, the following V–K iteration algorithm [25] is suggested:

- *Initialization*: Select a set of controller gains based on pole placement method or any other controller design scheme to predetermine a set of initial controller gains.
- *Step V*: Given the set of fixed controller gains K_i, $i\in I$, solve the following optimization problem

  $$\min_{Q_i,S_i,U_i,T_{ij}}\gamma_i$$
  subject to (24), $\Lambda_i-\gamma_i\mathrm{I}<0$ and $\Lambda_{ij}-\gamma_i\mathrm{I}<0$

  for a set of matrices Q_i, $i\in I$.
- *Step K*: Given the set of fixed controller gains Q_i, $i\in I$, solve the following optimization problem

  $$\min_{K_i,S_i,U_i,T_{ij}}\gamma_i$$
  subject to (24), $\Lambda_i-\gamma_i\mathrm{I}<0$ and $\Lambda_{ij}-\gamma_i\mathrm{I}<0$

  for a set of matrices K_i, $i\in I$.

  The algorithm continues till $\gamma_i<0$, $i\in I$.

Remark 4. Generalization of the results presented in this paper to the case of piecewise affine (PWA) dynamics is straightforward. This can be simply realized by augmenting the corresponding system matrices as demonstrated in [8]. The interested reader can refer to Appendix B.

5. Simulation results

In this section, we demonstrate the performance of the proposed approach using numerical examples. Example 1 deals with a switched system with Filippov solutions which is asymptotically stable at the origin; but, the disturbance attenuation performance is not satisfactory. Unlike Example 1, Example 2 considers an unstable PWL system in which both asymptotic stability and disturbance mitigation are investigated based on the proposed approach. Not to mention that in both cases uncertainties are also associated with the nominal systems.

5.1. Example 1

Suppose the state-space $X=\mathbb{R}^2$ is divided into four polytopes corresponding to the four quadrants of the second dimensional Euclidean space; i.e,

$$X_1=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_1>0\text{ and }x_2>0\}$$

$$X_2=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_1<0\text{ and }x_2>0\}$$

$$X_3=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_1<0\text{ and }x_2<0\}$$

$$X_4=\{(x_1,x_2)\in\mathbb{R}^2\,|\,x_1>0\text{ and }x_2<0\}\quad(27)$$

Consider a PWL system with Filippov solutions characterized by (19) and (20) where the associated system matrices are given by

$$A_1=A_3=\begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix},\quad A_2=A_4=\begin{bmatrix}0 & 1\\ 0.5 & 0\end{bmatrix}$$

$$B_1=B_2=B_3=B_4=\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$C_1=C_2=C_3=C_4=\begin{bmatrix}2\\ 0\end{bmatrix}^T$$

$$D_1=D_2=D_3=D_4=\begin{bmatrix}0\\ 0.1\end{bmatrix}^T$$

and the uncertainty bounds specified as

$$\mathcal{A}_1=\mathcal{A}_3=\begin{bmatrix}0 & 0.03\\ -0.03 & 0\end{bmatrix},\quad\mathcal{A}_2=\mathcal{A}_4=\begin{bmatrix}0.03 & 0\\ 0 & -0.03\end{bmatrix}$$

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$$\mathcal{C}_1 = \mathcal{C}_3 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}^T, \quad \mathcal{C}_2 = \mathcal{C}_4 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}^T$$

The matrices regarding the polytopes can be constructed as

$$E_1 = -E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} E_1 \\ \mathrm{I} \end{bmatrix}, \quad F_2 = \begin{bmatrix} E_2 \\ \mathrm{I} \end{bmatrix}, \quad F_3 = \begin{bmatrix} E_3 \\ \mathrm{I} \end{bmatrix}, \quad F_4 = \begin{bmatrix} E_4 \\ \mathrm{I} \end{bmatrix}$$

$$N_{12} = N_{34} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N_{14} = N_{23} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H_{12} = -H_{34} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H_{14} = -H_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Based on Theorem 2, a switching controller as defined in (17) is designed in order to ensure that (in addition to preserving the asymptotic stability property of the system) under zero initial conditions the disturbance signal of $v(t) = 5\cos(\pi t)$ is attenuated with $\eta = 0.05$. In this experiment, the constant scalars were preset to $\epsilon_{12} = \epsilon_{23} = \epsilon_{14} = \epsilon_{34} = 1$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 5$. The algorithm was initialized using pole placement method with initial pole positions of $(-1, -2)$ and controller gains of

$$K_1 = K_3 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}^T, \quad K_2 = K_4 = \begin{bmatrix} -3 \\ -5 \end{bmatrix}^T$$

The following solutions was obtained in two iterations

$$Q_1 = Q_3 = \begin{bmatrix} 78.29 & 5.96 \\ 5.96 & 3.01 \end{bmatrix}, \quad Q_2 = Q_4 = \begin{bmatrix} 33.06 & -1.35 \\ -1.35 & 65.14 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -0.9014 \\ -0.8292 \end{bmatrix}^T, \quad K_2 = K_4 = \begin{bmatrix} -0.1137 \\ -0.2715 \end{bmatrix}^T$$

$$\gamma_{min} = -7.03921 \times 10^{-4}$$

Fig. 1 sketches the trajectories of the closed-loop system without disturbance when the H_∞ controller is incorporated. This demonstrates that the Filippov solutions of the closed-loop system are asymptotically stable at **0**. One should observe that the solutions entering the facet $x_2 = 0$ cannot leave the facet (the so called attractive sliding mode property). This is due to the fact that the velocities at both regions X_1 and X_2 are toward the facet.

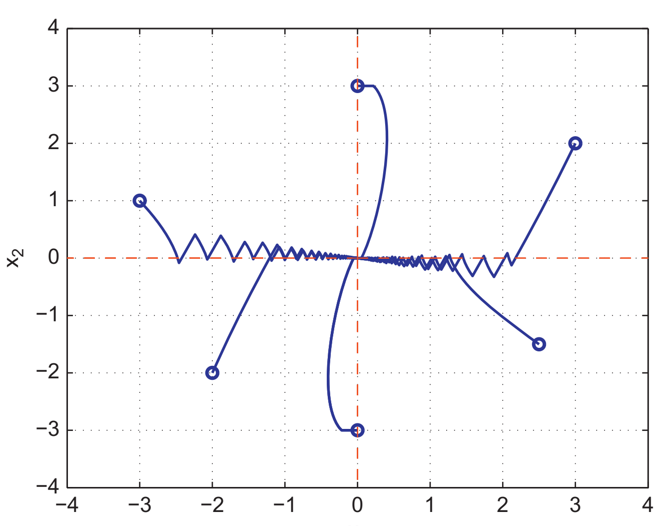

Fig. 1. The trajectories of the closed loop system. The dashed lines illustrate the facets.

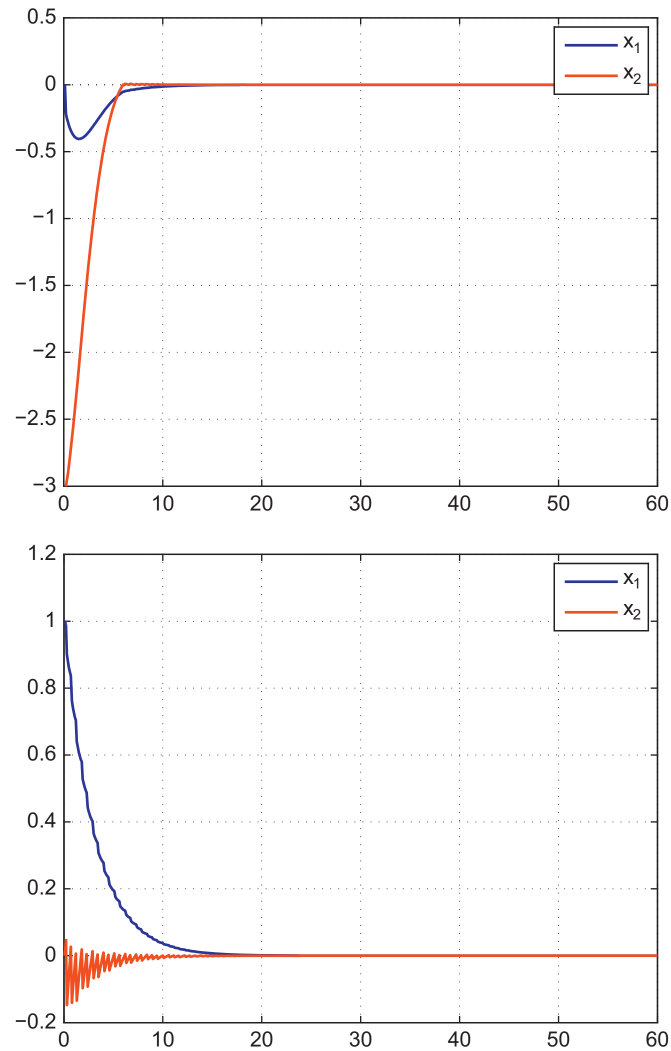

Fig. 2. Evolution of system states when the H_∞ controller is applied: with an initial condition on a non-attractive facet (top) and with an initial condition on an attractive facet (bottom).

We emphasize that this result could not been achieved by previous studies which excluded those solutions with infinite switching in finite time. Moreover, Fig. 2. displays the evolution of the states of the closed-loop system. The applied control inputs corresponding to simulations portrayed in Fig. 2 are available in Fig. 3. As can be inspected from Fig. 3 (and of course as expected), the control signals are discontinuous since switching occurs in the neighborhood of the attractive facets. This switching in the applied control signals diminishes considerably as the trajectories converge to origin in approximately 30 s.

The disturbance mitigation performance of the proposed method can also be deduced from Fig. 4. It can be discerned from the figure that the disturbance signal is considerably extenuated as the H_∞ controller is employed.

5.2. Example 2

For the sake of comparison, the example used in [6] is selected; but, instead of Carathéodory solutions, Filippov solutions are investigated. Therefore, the system structure has to be modified as delineated next. Consider an uncertain PWL system described by (19) and (20) with $I = \{1, 2, 3, 4\}$ and the state-space is a polyhedral set divided into four polytopes. The associated

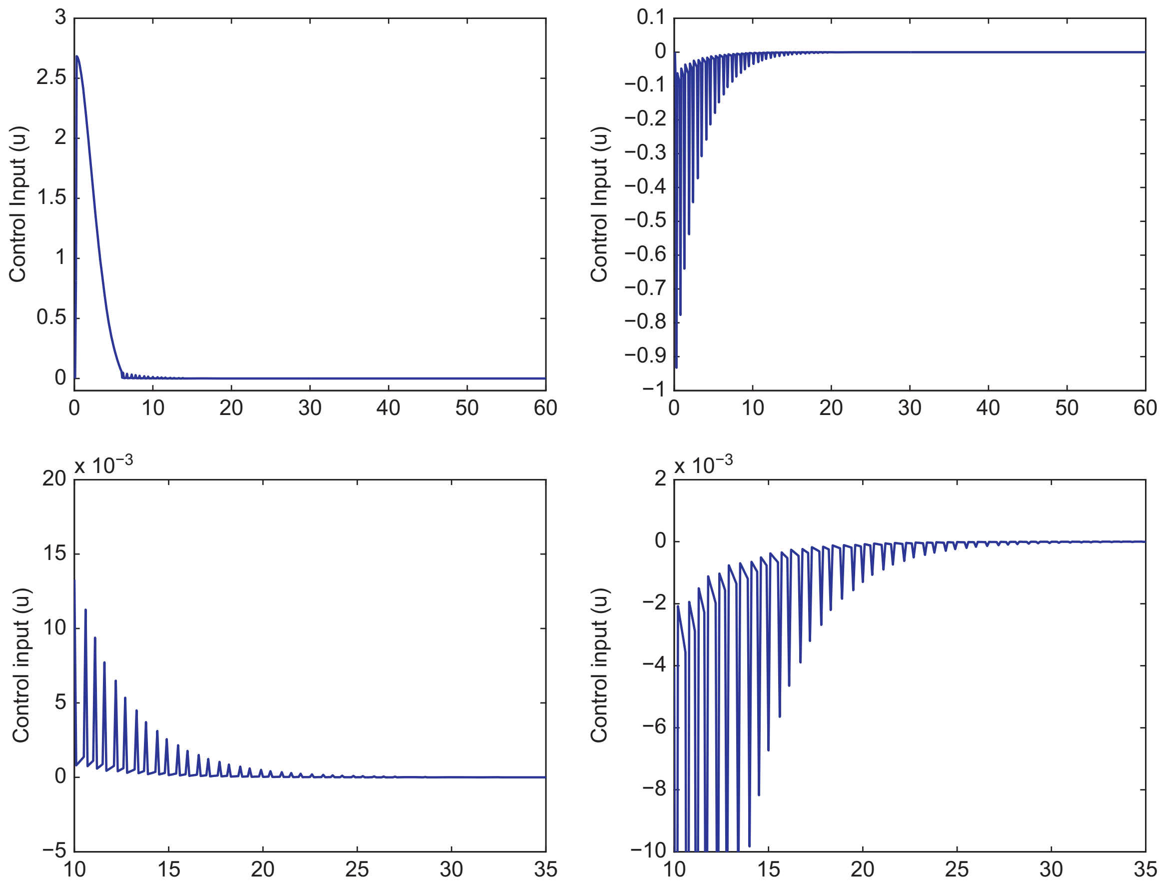

Fig. 3. Time histories of the applied control inputs corresponding to state evolutions provided in Fig. 2 (top), and the same figures enlarged (bottom).

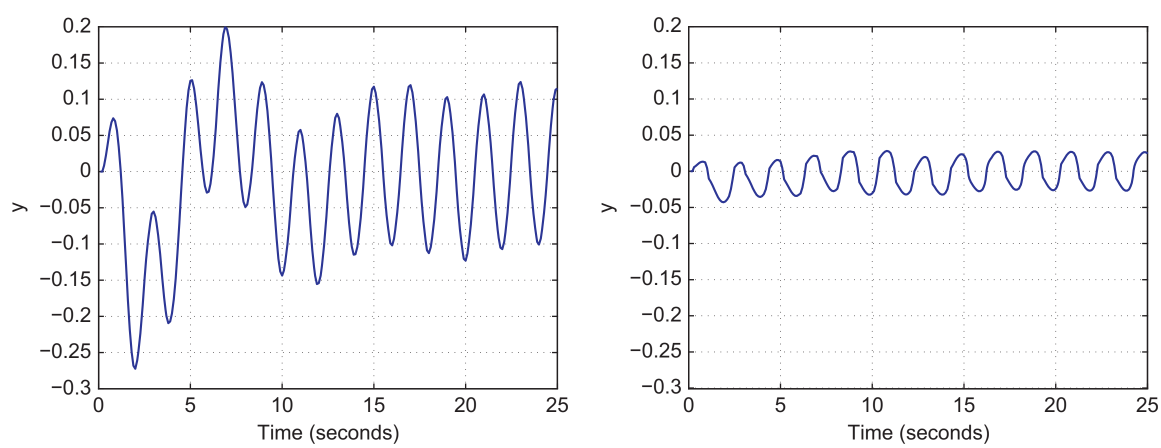

Fig. 4. Response of the closed loop control system with disturbance and zero initial condition: before applying the H_∞ controller (left) and after utilizing the H_∞ controller (right).

system matrices are

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

The uncertainty bounds are characterized as

$$\mathcal{A}_1 = \mathcal{A}_3 = \begin{bmatrix} 0 & 0.02 \\ -0.01 & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \mathcal{A}_4 = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.02 \end{bmatrix}$$

$$\mathcal{B}_1 = \mathcal{B}_3 = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \quad \mathcal{B}_2 = \mathcal{B}_4 = \begin{bmatrix} 0.02 \\ 0 \end{bmatrix}$$

The matrices characterizing the polytopes are given as follows

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_1=\begin{bmatrix}E_1\\ \mathrm{I}\end{bmatrix},\quad F_2=\begin{bmatrix}E_2\\ \mathrm{I}\end{bmatrix},\quad F_3=\begin{bmatrix}E_3\\ \mathrm{I}\end{bmatrix},\quad F_4=\begin{bmatrix}E_4\\ \mathrm{I}\end{bmatrix}$$

$$N_{12}=N_{34}=\begin{bmatrix}1\\ 1\end{bmatrix},\quad N_{14}=N_{23}=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$H_{12}=-H_{34}=\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix},\quad H_{14}=-H_{23}=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$$

It is worth noting that the open-loop system is unstable and since solutions with infinite switching in finite time are present the approach reported in [6] and common Lyapunov based methods are not applicable. The V–K iteration algorithm is initialized using pole placement method. The assigned closed-loop poles for the dynamics in each polytope are $(-3,-2)$ and the corresponding initial controller gains are

$$K_1=K_3=\begin{bmatrix}-119.5\\ -7\end{bmatrix}^T,\quad K_2=K_4=\begin{bmatrix}-5\\ 19.5\end{bmatrix}^T$$

Using the scheme presented in this paper for a set of constants $\epsilon_{12}=\epsilon_{23}=\epsilon_{14}=\epsilon_{34}=10$ and $\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=100$, the following solutions has been obtained

$$Q_1=Q_3=\begin{bmatrix}463.75 & 24.94\\ 24.94 & 2.39\end{bmatrix},\quad Q_2=Q_4=\begin{bmatrix}52.26 & -7.39\\ -7.39 & 763.47\end{bmatrix}$$

$$K_1=K_3=\begin{bmatrix}-637.72\\ -30.14\end{bmatrix}^T,\quad K_2=K_4=\begin{bmatrix}-21.53\\ -1.69\end{bmatrix}^T$$

$$\gamma_{min}=-2.7186\times 10^{-5}$$

for the H_∞ controller design with $\eta=0.1$ in five iterations. Consequently, it follows from Theorem 2 that the closed loop control system is asymptotically stable at the origin and the disturbance attenuation criterion is satisfied. Fig. 5. demonstrates the simulation results of four different initial conditions (in the absence of disturbance) using both the method expounded in [6] and the suggested scheme which prove the stability of the closed loop systems. It can be examined that the approach in [6] neglects the sliding motion along the facets; hence, it cannot take into account the solutions with infinite switching in finite time which are intrinsic to switched systems. Notice, in particular, that solutions with infinite switching in finite time on facets are also present (see Fig. 6.) when the proposed method is exploited. Additionally, the applied control inputs associated with the simulations in Fig. 6 are shown in Fig. 7. Once again as expected, the controller starts to switch (discontinuous

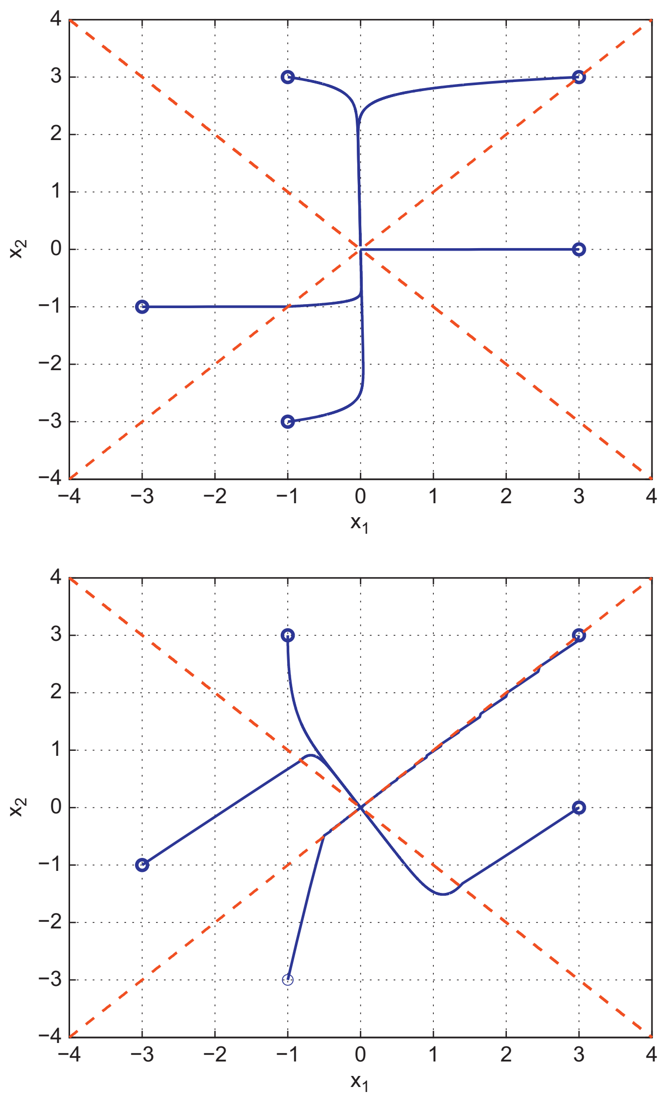

Fig. 5. The trajectories of the closed loop control system. H_∞ synthesis based on [6] (top), and using the proposed methodology (bottom). The dashed lines illustrate the facets.

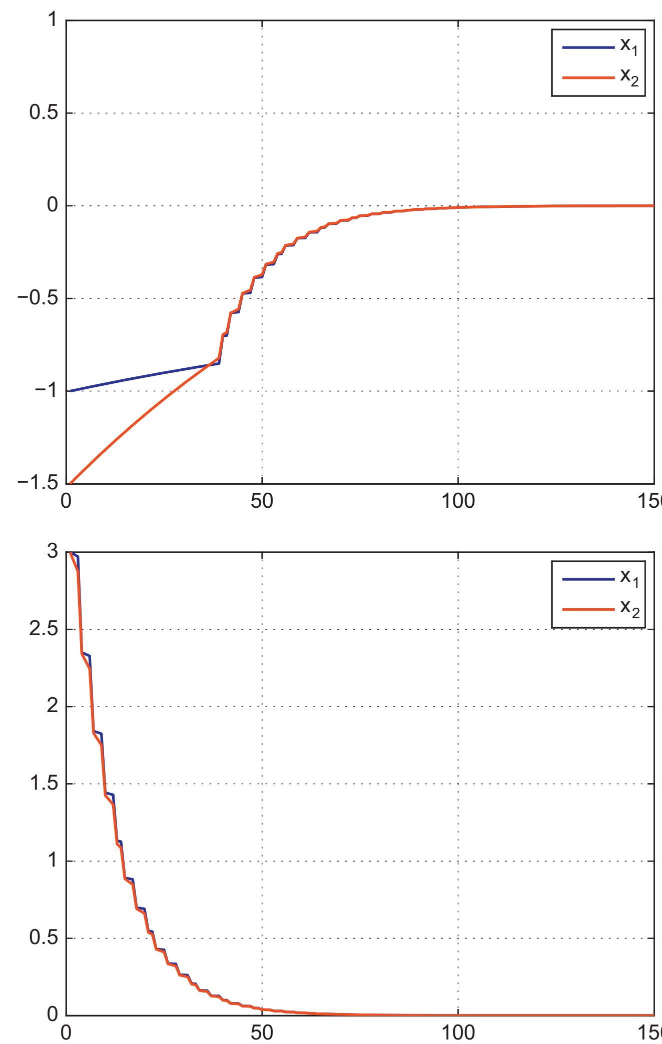

Fig. 6. Evolution of system states when the H_∞ controller is applied: with an initial condition in the interior of a polytope (top) and with an initial condition on an attractive facet (bottom).

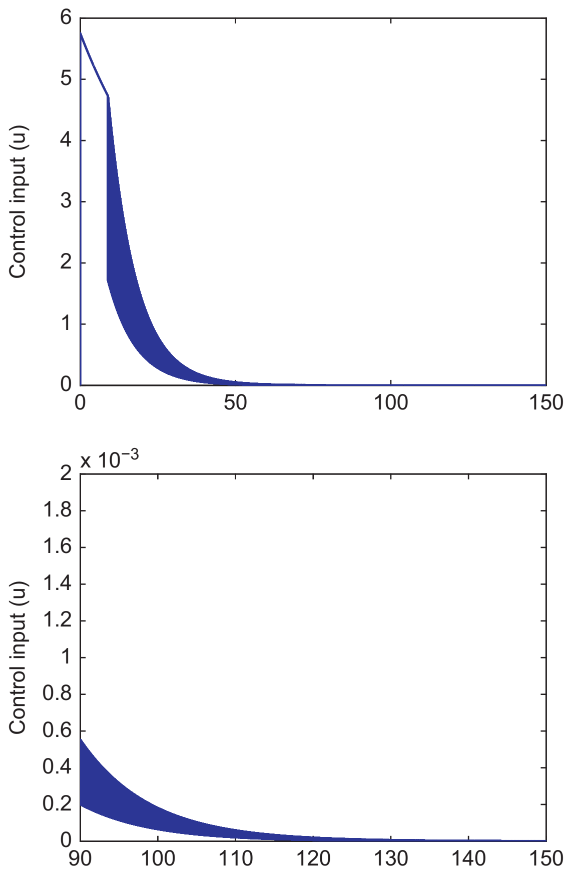

Fig. 7. Time histories of the applied control inputs corresponding to state evolutions provided in Fig. 6 (top), and the same figures enlarged (bottom).

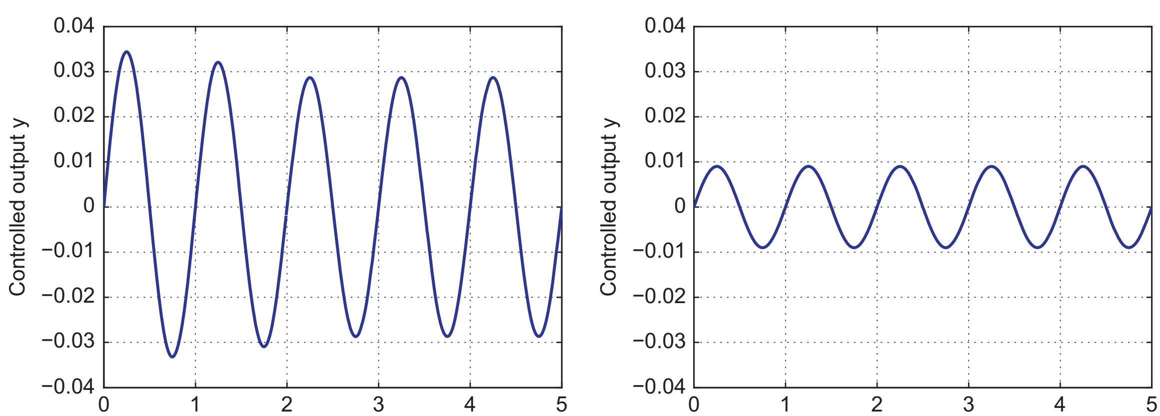

Fig. 8. Response of the closed loop control system with disturbance and zero initial condition: the stable controller synthesis (left) and the H_∞ controller synthesis (right).

behavior) when the trajectories reach an attractive facet. The fluctuations of the control signal mitigates significantly in around 130 s, implying that the solutions have converged to the origin. This should be opposed to the results in [6] where only Carathódory solutions are taken into account. Additionally, the simulation results in the presence of disturbance ($\nu(t) = 4\,\sin(2\pi t)$) and zero initial conditions are illustrated in Fig. 8, which ascertains the disturbance attenuation performance of the proposed controller.

6. Conclusions

In this paper, the stability and control problem of uncertain PWL systems with Filippov Solutions was considered. The foremost purpose of this research was to extend the previous results on switched systems defined on polyhedral sets to the case of solutions with infinite switching in finite time and sliding motions. In this regard, a set of matrix inequality conditions are brought forward to investigate the stability of a PWL system in the framework of Filippov solutions. Additionally, a method based on BMIs are devised for the synthesis of stable and robust H_∞ controllers for uncertain PWL systems with Filippov solutions. These schemes has been examined through simulation experiments. The following subjects are suggested for further research:

- Due to practical considerations, it is sometimes desirable to assuage the switching frequency in the control signal which may contribute to unwanted outcomes, e.g., high heat losses in

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power circuits, mechanical wear, and etc. The authors suggest the application of *chattering-free* techniques such as boundary layer control (BLC). However, how these methods could be embedded in the framework of the analyses presented here is still an open problem.

- The robust control and stability results can be extended to other solution types for discontinuous systems, e.g. Krasovskii, Aizerman and Gantmakher (see the expository review [21]).
- The curious reader may consult [26] for novel (robust) stability analysis results on nonlinear switched systems defined on compact sets in the context of Filippov solutions.

Appendix A. Proof of Theorem 1

From (21)–(22) and Lemma 2, it can be discerned that the Filippov solutions of the closed loop system (19) converge to **0** asymptotically. Additionally, since $Q_i = F_i^T M F_i$ and $F_i x = F_j x$, for all $x \in X_i \cap X_j$ the continuity of the Lyapunov functions is assured. What remains is to show that the disturbance attenuation performance is η. Define a multi-valued function

$$\Upsilon(x) = \{z \in \mathbb{R} \,|\, z = V_i(x) \quad \text{if } x \in X_i\} \tag{A.1}$$

and set $\Gamma(x) = co(\Upsilon(x))$. This can be thought of as a switched Lyapunov function. Differentiating and integrating Γ with respect to t yields

$$\begin{aligned}\int_0^\infty \frac{d\Gamma}{dt}\,dt &= \int_0^{t_1} [x^T(A_{c1}^T Q_1 + Q_1 A_{c1})x + \upsilon^T D_{c1}^T Q_1 x + x^T Q_1 D_{c1}\upsilon]\,dt + \cdots \\ &+ \int_{t_1}^{t_2} [x^T(A_{c2}^T Q_2 + Q_2 A_{c2})x + \upsilon^T D_{c2}^T Q_2 x + x^T Q_2 D_{c2}\upsilon]\,dt + \cdots \\ &+ \sum_{j=1}^{r} \alpha_j \left\{ \int_{t_{k-1}}^{t_k} [x^T(A_{cj}^T Q_k + Q_k A_{cj})x + \upsilon^T D_{cj}^T Q_k x + x^T Q_k D_{cj}\upsilon]\,dt \right\} + \cdots \\ &+ \sum_{j=1}^{m} \beta_j \left\{ \int_{t_{l-1}}^{t_l} [x^T(A_{cj}^T Q_l + Q_l A_{cj})x + \upsilon^T D_{cj}^T Q_l x + x^T Q_l D_{cj}\upsilon]\,dt \right\} + \cdots \\ &+ \int_{t_n}^{\infty} [x^T(A_{cn}^T Q_n + Q_n A_{cn})x + \upsilon^T D_{cn}^T Q_n x + x^T Q_n D_{cn}\upsilon]\,dt\end{aligned}$$

wherein $\alpha_j, \beta_j > 0$ such that $\sum_{j=1}^n \alpha_j = 1$, and $\sum_{j=1}^n \beta_j = 1$. m and r are the number of neighboring cells to a boundary where the solutions possess infinite switching in finite time (in the time intervals of $[t_{k-1}, t_k]$ and $[t_{l-1}, t_l]$), respectively. With the above formulation, we consider a state evolution scenario including the interior of different polytopes as well as the facets. Suppose conditions (22) and (23) hold, then it follows that

$$\begin{aligned}&\int_a^b [x^T(A_{ci}^T Q_i + Q_i A_{ci})x + \upsilon^T D_{ci}^T Q_i x + x^T Q_i D_{ci}\upsilon]\,dt \\ &< \int_a^b [x^T(-E_i^T U_i E_i - \eta^{-2} Q_i D_{ci} D_{ci}^T Q_i - C_{ci}^T C_{ci})x \\ &\quad + \upsilon^T D_{ci}^T Q_i x + x^T Q_i D_{ci}\upsilon + \eta^2 \upsilon^T \upsilon - \eta^2 \upsilon^T \upsilon]\,dt \\ &\le \int_a^b [-y^T y + \eta^2 \upsilon^T \upsilon - \eta^2(\upsilon - \eta^{-2} D_{ci}^T Q_i x)^T(\upsilon - \eta^{-2} D_{ci}^T Q_i x))]\,dt \\ &\le \int_a^b (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt\end{aligned}$$

Correspondingly,

$$\begin{aligned}&\sum_{j=1}^{n} \alpha_j \left\{ \int_c^d [x^T(A_{cj}^T Q_i + Q_i A_{cj})x + \upsilon^T D_{cj}^T Q_i x + x^T Q_i D_{cj}\upsilon]\,dt \right\} \\ &< \sum_{j=1}^{n} \alpha_j \left\{ \int_c^d [x^T(-W_{ij} N_{ij}^T - N_{ij} W_{ij}^T - H_{ij}^T T_{ij} H_{ij} \right. \\ &\quad \left. - \eta^{-2} Q_i D_{cj} D_{cj}^T Q_i - C_{cj}^T C_{cj})x + \eta^2 \upsilon^T \upsilon - \eta^2 \upsilon^T \upsilon]\,dt \right\} \\ &\le \int_c^d \left(-y^T y + \eta^2 \upsilon^T \upsilon - \sum_{j=1}^{n} \alpha_j (\eta^2(\upsilon - \eta^{-2} D_{cj}^T Q_i x)^T(\upsilon - \eta^{-2} D_{cj}^T Q_i x)) \right) dt \\ &\le \int_c^d (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt\end{aligned}$$

where, $a,b,c,d > 0$ are arbitrary non-negative constants ($b > a$, and $d > c$). Finally, we arrive at the justification that

$$\begin{aligned}\int_0^\infty \frac{d\Gamma}{dt}\,dt &\le \int_0^{t_1} (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt + \int_{t_1}^{t_2} (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt + \cdots \\ &+ \int_{t_{k-1}}^{t_k} (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt + \cdots \\ &+ \int_{t_{l-1}}^{t_l} (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt + \cdots \\ &+ \int_{t_n}^{\infty} (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt\end{aligned}$$

which reduces to

$$\Gamma(x(\infty)) - \Gamma(x(0)) \le \int_0^\infty (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt \tag{A.2}$$

Moreover, note that $x(\infty) = x(0) = \mathbf{0}$. This can be concluded from the assumption on zero initial conditions, and from the fact that the system is asymptotically stable at origin (as demonstrated earlier in this proof). Consequently, we have

$$0 \le \int_0^\infty (-y^T y + \eta^2 \upsilon^T \upsilon)\,dt \tag{A.3}$$

which is equivalent to (18). This completes the proof.

Appendix B. Generalization to piecewise affine systems

It is worth noting that all results obtained for PWL systems with Filippov solutions can also be accommodated for PWA systems. However, the following modifications should be applied in advance.

Consider a PWA system with Filippov solutions $\overline{\mathcal{S}} = \{\overline{\mathcal{X}}, \mathcal{U}, \mathcal{V}, \overline{X}, I, \overline{F}, \overline{G}\}$, where $\overline{\mathcal{X}} \subseteq \mathbb{R}^{n+1}$ is a polyhedral set representing the state space, $\overline{X} = \{\overline{X_i}\}_{i \in I}$ is the set containing the polytopes in $\overline{\mathcal{X}}$ with index set $I = \{1, 2, \ldots, n_X\}$ (note that $\bigcup_{i \in I} \overline{X_i} = \overline{\mathcal{X}}$). $\overline{F} = \{\overline{f_i}\}_{i \in I}$ and $\overline{G} = \{\overline{g_i}\}_{i \in I}$ are families of linear functions associated with the (augmented) system states $\overline{x} = [x^T \ 1]^T \in \overline{\mathcal{X}}$ and outputs y. Each $\overline{f_i}$ consists of six elements $(\overline{A_i}, \overline{B_i}, \overline{D_i}; \Delta\overline{A_i}, \Delta\overline{B_i}, \Delta\overline{D_i})$ and each $\overline{g_i}$ is composed of four elements $(\overline{C_i}, G_i; \Delta\overline{C_i}, \Delta G_i)$.

The following augmented system matrices can be constructed

$$\overline{A_i} = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \overline{B_i} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \overline{D_i} = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \quad \overline{C_i} = [C_i \ \ 0] \tag{B.1}$$

where a_i is the affine term associated with the dynamics in the polytope $\overline{X_i}$, and correspondingly the augmented uncertain matrices

$$\Delta\overline{A_i} = \begin{bmatrix} \Delta A_i & \Delta a_i \\ 0 & 0 \end{bmatrix}, \quad \Delta\overline{B_i} = \begin{bmatrix} \Delta B_i \\ 0 \end{bmatrix},$$

$$\Delta\overline{D_i} = \begin{bmatrix} \Delta D_i \\ 0 \end{bmatrix}, \quad \Delta\overline{C_i} = [\Delta C_i \ \ 0] \tag{B.2}$$

Besides, the matrices characterizing each polytope can also be modified as

$$\overline{E_i} = [E_i \ \ e_i] \quad \text{and} \quad \overline{F_i} = [F_i \ \ f_i] \tag{B.3}$$

Then, each polytope is defined as

$$\overline{X_i} = \{\overline{x} \in \overline{\mathcal{X}} \,|\, \overline{E_i}\overline{x} \succcurlyeq 0\} \tag{B.4}$$

and for all $\overline{x} \in \overline{X_i} \cap \overline{X_j}$ and $(i,j) \in \tilde{I}$ it holds that

$$\overline{F_i}\overline{x} = \overline{F_j}\overline{x} \tag{B.5}$$

Hence, the candidate Lyapunov function in effect in the polytope $\overline{X_i}$ is formulated as

$$V_i(x) = \overline{x}^T \overline{F_i}^T \overline{M}\, \overline{F_i}\overline{x} = \overline{x}^T \overline{Q_i}\overline{x} \tag{B.6}$$

It is possible that the facets may not have intersections at the origin. Therefore, let

$$\overline{N}_{ij} = \begin{bmatrix} N_{ij} \\ n_{ij} \end{bmatrix} \quad \text{and} \quad \overline{H}_{ij} = \begin{bmatrix} H_{ij} & h_{ij} \\ 0 & 0 \end{bmatrix} \tag{B.7}$$

and each boundary can be characterized as

$$\overline{X_i} \cap \overline{X_j} = \{\overline{x} \in \overline{\mathcal{X}} \,|\, \overline{N}_{ij}^T \overline{x} \approx 0,\ \overline{H}_{ij}\overline{x} \succcurlyeq 0,\ (i,j) \in \tilde{I}\} \tag{B.8}$$

With the above formulation at hand, one can utilize the results given in this paper for PWA systems. This is simply accomplished by replacing the associated matrices for PWL systems by their PWA counterparts.

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