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Robust Stability Analysis of Nonlinear Switched Systems with Filippov Solutions

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Abstract: This paper addresses the stability problem of a class of nonlinear switched systems with partitioned state-space and state-dependent switching. In lieu of the Carathéodory solutions, the general Filippov solutions are considered. This encapsulates solutions with infinite switching in finite time. Based on the theory of differential inclusions, a Lyapunov stability theorem is brought forward. These results are also extended to switched systems subject to polytopic uncertainty. Furthermore, the proposed stability theorems are reformulated using the sum of squares decomposition method which provides sufficient means to construct the corresponding Lyapunov functions via available semi-definite programming techniques.

1. INTRODUCTION

A large group of engineering applications give rise to systems which encompass both discrete and continuous dynamics. Mathematically, these systems are characterized by a collection of indexed differential or difference equations describing each subsystem and a switching rule between them. This rich family of systems is referred to switched or more generally hybrid systems. Examples of such systems in real world have been studied in open literature e.g. Wisniewski and Larsen [2008] and Larsen et al. [2007].

However despite numerous applications, their stability analysis has not been covered completely (Lin and Antaklis [2009]). Several interesting phenomena arise when dealing with such systems; namely, even if all the subsystems are exponentially stable, one cannot guarantee the stability of the overall system (Branicky [1998]). Conversely, an appropriate switching law may contribute to stability even when all subsystems are unstable (Liberzon [2003]). Still, depending on the considered type of solutions (Carathéodory, Filippov, and etc.), discrepant stability phenomenon may follow. As an illustration, a switched system with stable Carathéodory solutions, may possess divergent Filippov solutions (see Leth and Wisniewski [2012]).

This research is mainly motivated by two contributions (Pranja and Papachristodoulou [2003]) and (Leth and Wisniewski [2012]). Leth and Wisniewski [2012] exploited the theory of differential inclusions (DI) and suggested Lyapunov-like stability theorem for piece-wise linear switched systems defined on polyhedral sets with Fillipov solutions. Pranja and Papachristodoulou [2003] proposed sum of squares based stability analysis tools for a class of hybrid systems; however, a unified stability theorem has not been suggested or fully established. Additionally, the solutions considered in the latter article are in the sense of Carathéodory which connotate the exclusion of solutions with infinite switching in finite time from the analysis. In the present paper, firstly, we extend the results reported in (Leth and Wisniewski [2012]) to the general nonlinear switched systems defined on regular sets by incorporating the theoretical notions of differential inclusions. Secondly, the robust stability problem of switched systems with polytopic uncertainty and Filippov solutions is addressed. Lastly, we propose sufficient conditions based on sum of squares (SOS) decomposition for the suggested stability theorems. This ensures computationally efficient means to investigate the stability of switched systems.

This paper is organized as follows. In the next section, the notations and some mathematical concepts adopted in this study are limned. The main results of this paper are brought forward in section 3. Section 4 demonstrates the accuracy of the proposed methodology via an example. Finally, section 5 concludes the paper.

2. MATHEMATICAL PRELIMINARIES

The notations employed in this paper are relatively straightforward. $\mathbb{R}_{\geq 0}$ denotes the set $[0,\infty)$. $\|\cdot\|$ denotes the Euclidean vector norm on $\mathbb{R}^n$, $\langle\cdot\rangle$ the inner product, and $\mathcal{B}^n_\epsilon$ the closed ball of radius ϵ in $\mathbb{R}^n$ centered at origin. $\mathcal{P}$ accounts for the set of polynomial functions $p:\mathbb{R}^n\to\mathbb{R}$ and $\mathcal{P}_{sos}\subset\mathcal{P}$ is the subset of polynomials with an SOS decomposition; i.e, $p\in\mathcal{P}_{sos}$ if and only if there are $p_i\in\mathcal{P}, i\in\{1,\ldots,k\}$ such that $p=p_i^2+\cdots+p_k^2$. In this study, we consider a class of n-dimensional

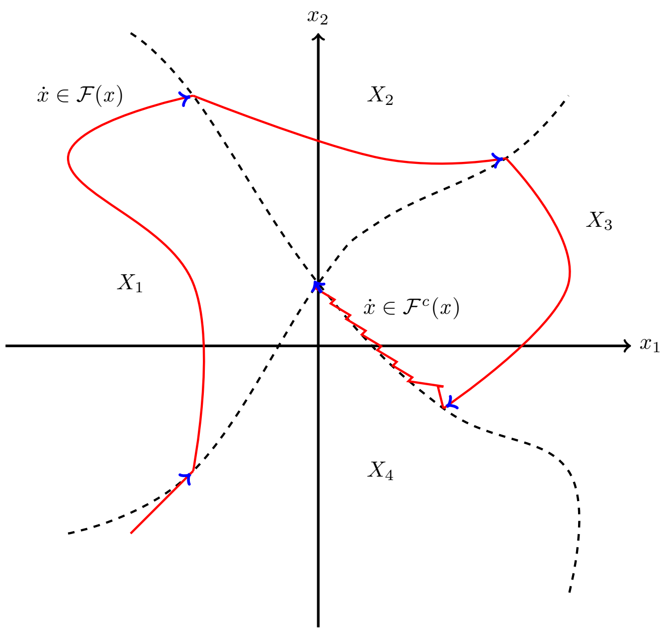

Fig. 1. The trajectories of a switched system. Notice that the motion follows either DI (1) or (2).

nonlinear switched systems $\mathcal{S} = \{G, \mathcal{X}, I, \mathscr{F}\}$ wherein G is compact and defines the state-space, $\mathcal{X} = \{X_i\}_{i \in I}$ is the set containing (closed) partitions of G with index set $I = \{1, 2, 3, ..., N\}$, and $\mathscr{F} = \{F_i\}_{i \in I}$ a family of smooth functions ($F_i : U_i \to \mathbb{R}^n$ with U_i an open neighborhood of X_i). Note that $G = \bigcup_{i \in I} X_i$, $X_i = int(X_i) \cup bdX_i$, $int(X_i) \neq \emptyset$ for all $X_i \in \mathcal{X}$, and $X_i \cap X_j \subset bdX_i \cup bdX_j$. Also, denote $\tilde{I} = \{(i, j) \in I \times I \mid X_i \cap X_j \neq \emptyset,\ i \neq j\}$ the set of index pairs which determines the partitions with non-empty intersections. Besides, each point of G has an open neighborhood intersecting only finitely many elements of $\mathcal{X}$ (local finiteness). We also posit that for any $x \in bdX_i$ and $v \in \mathbb{R}^n$ there is $a > 0$ and j such that $x + vt \in X_j$ for all $t \in [0, a)$ (characterization of a *nice* partitioning of the state space). Remark that partitioning by polyhedral sets assures that this latter property is satisfied. This is the case when considering switched systems defined on polyhedral sets, e.g. piecewise linear systems.

The global dynamics is described by the following differential inclusions

$$\dot{x}(t) \in \mathcal{F}(x(t)) \tag{1}$$

$$\dot{x}(t) \in \mathcal{F}^c(x(t)) \tag{2}$$

where the set-valued maps $\mathcal{F}$ and $\mathcal{F}^c$ are defined by

$$\mathcal{F} : G \to 2^G; x \mapsto \{v \in \mathbb{R}^n \mid v = F_i(x) \text{ if } x \in X_i\} \tag{3}$$

$$\mathcal{F}^c : G \to 2^G; x \mapsto co(\mathcal{F}(x)) \tag{4}$$

where 2^G denotes the power set of G, and the notation $co(\cdot)$ signifies the convex hull. The choice of whether the dynamics is modeled by (1) or (2) depends on the nature of the motion to be considered (see Fig. 1). Pertaining to the solutions of discontinuous and switched dynamical systems, the interested reader is referred to the didactic review in (Cortes [1998]).

In the sequel, we apply the following notions. For $Q \subset \mathbb{R}^n$, $T_Q(x)$ denotes the Bouligand's contingent cone of Q at $x \in Q$. If Q is convex then $T_Q(x)$ is closure of the cone spanned by $Q - \{x\}$. In addition, if x is in the interior of Q, we have $T_Q(x) = \mathbb{R}^n$ (Aubin and Cellina [1984]). The upper contingent derivative of a function $U : \mathbb{R}^n \to \mathbb{R}$ at x_0 in the direction v_0 is defined as

$$D_+U(x_0)(v_0) = \lim_{\substack{l \to 0^+ \\ v \to v_0}} \inf \left(\frac{U(x + lv) - U(x)}{l}\right) \tag{5}$$

Note in particular that, if $U(x)$ is Gâteaux differentiable then it holds that

$$D_+U(x_0)(v_0) = \langle \frac{\partial U}{\partial x}, v \rangle \tag{6}$$

where $\frac{\partial U}{\partial x}$ denotes the column vector with first-order partial derivatives of $U(x)$.

Proposition 1. The set-valued map defined by (3) is upper semi-continuous; i.e., for any $x \in G$ and any $\epsilon > 0$ there exists a $\sigma(\epsilon, x) \leq \epsilon$ such that $\forall \acute{x} \in x + \mathcal{B}^n_\sigma$, $\mathcal{F}(\acute{x}) \subset \mathcal{F}(x) + \mathcal{B}^n_\epsilon$. Correspondingly, $\mathcal{F}^c$ defined by (4) is an upper semi-continuous set valued map with non-empty, convex and compact values.

Proof. For all $x \in int(X_i)$, $i \in I$, $\mathcal{F}(x) = F_i(x)$ is a one point set and since each F_i is continuous, $\mathcal{F}$ is upper semi-continuous at any $x \in int(X_i)$. Furthermore, for any $x \in bdX_i \cup bdX_{i+1} \cup \cdots \cup bdX_j$, $\mathcal{F}(x) = \{F_i(x), F_{i+1}(x), \ldots, F_j(x)\}$ is a multi-valued set. Because each F_i is continuous, it follows that for all $x \in X_i$ and $\epsilon > 0$ there exists a $\sigma_i > 0$ such that $\forall \acute{x} \in x + \mathcal{B}^n_{\sigma_i}$, $F_i(\acute{x}) \in F_i(x) + \mathcal{B}^n_\epsilon$. To demonstrate that $\mathcal{F}$ is upper semi-continuous, it suffices to choose $\sigma = \min_i \sigma_i$. Additionally, because each of the maps F_i, $i \in I$, is continuous and $\mathcal{F}(x)$ is finite for all $x \in G$ (finiteness of $\mathcal{F}$ follows from partitioning with local finiteness property); then, from Lemma 16 in (p. 66, Filippov [1988]), it follows that $\mathcal{F}^c$ is also upper semi-continuous.

It is also worth noting that $\mathcal{F}$ cannot be lower semi-continuous at any point $x \in X_i \cap X_j$, $(i, j) \in \tilde{I}$, on a boundary, since $\mathcal{F}$ is not a one point set. For $T > 0$, let S_T denote either $[0, T)$ or $[0, T]$. By a Carathéodory solution of DI (1) at $\zeta_0 \in G$, we understand an absolutely continuous function $S_T \to G; t \mapsto \zeta(t)$ which solves the following Cauchy problem

$$\dot{\zeta}(t) \in \mathcal{F}(\zeta(t)) \quad \text{a.e.}, \quad \zeta(0) = \zeta_0 \tag{7}$$

A Filippov solution to DI (1) at $\zeta_0 \in G$ is a solution to (7) with $\mathcal{F}$ supplanted by $\mathcal{F}^c$ (Filippov [1988]).

We recall the following facts from the theory of differential inclusions. Let W be some non-negative function defined on $Graph(\mathcal{F}) = \bigcup_{x \in G} \{x\} \times G \subset G \times G$. We shall say that a function $\Phi : G \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for $\mathcal{F}$ with respect to W if for all $x \in G$ and some $v \in \mathcal{F}(x)$ the following "Lyapunov property" holds

$$D_+\Phi(x)(v) + W(x, v) \leq 0 \tag{8}$$

The following proposition which is derived from Corollary 1 and 2 in (p. 292, Aubin and Cellina [1984]) illustrates that, in fact, the existence of an equilibrium can be inferred from the Lyapunov property.

Proposition 2. If there exist a continuous non-negative function $\Phi : G \to \mathbb{R}_{\geq 0}$ and a positive definite function $W : Graph(\mathcal{F}^c) \to \mathbb{R}_{>0}$ satisfying (8) for all $x \in G$ and some $v \in \mathcal{F}^c(x)$, then there exists an equilibrium $x_* \in G$ for $\mathcal{F}^c$.

Indeed, if the conditions of Proposition 2 hold, then

$$\forall x \in G, \quad \mathcal{F}^c(x) \cap T_G(x) \neq \emptyset \tag{9}$$

Furthermore, if $F^c(G)$ is bounded, from Theorem 1 in (p. 180, Aubin and Cellina [1984]), it follows that at any point $x_0 \in G$, there exists a Filippov solution of (1) defined on $\mathcal{S}_\infty$ which remains in G (viability).

Now, we are ready to assert a stability condition for the set valued map $\mathcal{F}^c$ which is a direct result of applying Theorem 8.4 in (p. 176, Smirnov [2002]).

Proposition 3. Suppose $0 \in \mathcal{F}^c(0)$. If there exist $\epsilon > 0$ and continuous positive definite functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for each $x \in \mathcal{B}^n_\epsilon$

$$D_+V(x)(v) + W(x) \leq 0 \quad \text{for all} \quad v \in \mathcal{F}^c(x) \tag{10}$$

Then the equilibrium point 0 is asymptotically stable.

Recall that if there exists an SOS decomposition for $p(x)$, then it follows that $p(x)$ is non-negative. Unfortunately, the converse does not hold in general; that is, there exist non-negative polynomials which do not have an SOS decomposition. An epitome of this class of non-negative polynomials is the Motzkin's polynomial (Motzkin [1965]) given by

$$p(x) = 1 - 3x_1^2x_2^2 + x_1^2x_2^4 + x_1^4x_2^2 \tag{11}$$

which is non-negative for all $x \in \mathbb{R}^2$. This imposes, more or less, some sort of conservatism when utilizing SOS based methods. The next proposition gives an interesting formulation to the SOS decomposition problem.

Proposition 4. (Choi et al. [1995]). A polynomial $p(x)$ of degree $2d$ belongs to $\mathcal{P}_{sos}$ if and only if there exist a positive semi-definite matrix Q (known as the Gram matrix) and a vector of monomials $Z(x)$ which contains all monomial of x of degree $\leq d$ such that $p(x) = Z^T(x)QZ(x)$.

Chesi et al. [1999] evinced that testing whether a polynomial is SOS can be formulated as a set of LMI feasibility tests. Subsequently, Parrilo [2003] demonstrated that the answer to the query that whether a given polynomial $p(x)$ is SOS or not can be investigated via semi-definite programming methodologies.

Proposition 5. (Parrilo [2003]). Given a finite set $\{p_i\}_{i=0}^m \in \mathcal{P}$, the existence of a set of scalars $\{a_i\}_{i=1}^m \in \mathbb{R}$ such that

$$p_0 + \sum_{i=1}^{m} a_i p_i \in \mathcal{P}_{sos} \tag{12}$$

is an LMI feasibility problem.

The subsequent proposition formalizes the problem of constrained positivity of polynomials which is a direct result of applying Positivstellensatz method (Stengle [1994]).

Proposition 6. (Chesi [2010]). Let $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^l$ belong to $\mathcal{P}$, then

$$\begin{aligned} p(x) \geq 0 \quad & \forall x \in \mathbb{R}^n : a_i(x) = 0, \forall i = 1, 2, ..., k \\ & \text{and} \quad b_j(x) \geq 0, \forall j = 1, 2, ..., l \end{aligned} \tag{13}$$

is satisfied, if the following holds

$$\exists r_1, r_2, \ldots, r_k \in \mathcal{P} \quad \text{and} \quad \exists s_0, s_1, \ldots, s_l \in \mathcal{P}_{sos}$$

$$p = \sum_{i=1}^{k} r_i a_i + \sum_{i=1}^{l} s_i b_i + s_0 \tag{14}$$

Proposition 7. The multivariable polynomial $p(x)$ is strictly positive ($p(x) > 0 \quad \forall x \in \mathbb{R}^n$), if there exists a $\lambda > 0$ such that

$$\big(p(x) - \lambda\big) \in \mathcal{P}_{sos} \tag{15}$$

At this point, we are prepared to delineate the main contributions of this paper.

3. MAIN RESULTS

In this section, we will incorporate the mathematical notions given in the previous section to derive asymptotic stability conditions for the class of nonlinear switched systems with Fillipov solutions defined on regular sets. Then, we present a theorem for robust asymptotic stability of switched systems with polytopic uncertainty. Finally, we bring forward sufficient conditions for stability using SOS techniques.

3.1 Asymptotic Stability Conditions for Switched Systems

Consider the switched system $\mathcal{S}$ and let (2) describe the Filippov solutions of $\mathcal{S}$. It is assumed that 0 is an interior point of G, and that it is located on some boundary of partitions. Note that $0 \in \mathcal{F}^c(0)$, hence 0 is an equilibrium. Denote by $\{V_i(x)\}_{i\in I}$ a family of positive definite and continuously differentiable ($\mathcal{C}^1$) functions ($V_i : X_i \to \mathbb{R}_{\geq 0}$). We define a set valued map $\Psi(x)$ as

$$\Psi : G \to 2^{\mathbb{R}}; x \mapsto \{z \in \mathbb{R} \mid z = V_i(x) \quad \text{if} \quad x \in X_i\} \tag{16}$$

Obviously, $\Psi(x)$ can be considered as a switched system type of a candidate Lyapunov function.

Proposition 8. If $V_i(x) = V_j(x)$ for all $x \in X_i \cap X_j$ and all $(i, j) \in \tilde{I}$, then $\Psi(\cdot)$ is real single-valued ($\Psi : G \to \mathbb{R}$) and locally Lipschitzean.

Notice that, Proposition 8 does not impose any restrain on the structure of $\{V_i(x)\}_{i\in I}$, e.g. homogenous or quadratic forms as was done in Leth and Wisniewski [2012]. This considerably mitigates the conservatism in finding the family of Lyapunov functions $\{V_i(x)\}_{i\in I}$.

Proposition 9. Suppose

I) $\big\langle \frac{\partial V_i(x)}{\partial x}, F_i(x)\big\rangle < 0$ for all $x \in X_i \setminus \{0\}$ and all $i \in I$,
II) $\big\langle \frac{\partial V_i(x)}{\partial x}, F_j(x)\big\rangle < 0$ for all $x \in X_i \cap X_j \setminus \{0\}$ and all $(i, j) \in \tilde{I}$.

Then there exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

III) $\big\langle \frac{\partial V_i(x)}{\partial x}, F_i(x)\big\rangle < -W(x)$ for all $x \in X_i \setminus \{0\}$ and all $i \in I$,
IV) $\big\langle \frac{\partial V_i(x)}{\partial x}, F_j(x)\big\rangle < -W(x)$ for all $x \in X_i \cap X_j \setminus \{0\}$ and all $(i, j) \in \tilde{I}$.

Proof. Suppose for each X_i, with $i \in I$, there exist an open neighborhood T_i of X_i such that condition (I) holds due to the compactness of X_i. Then, the collection of such open neighborhoods $\{T_i\}_{i\in I}$ is an open cover of G such that $G \subseteq \bigcup_{i\in I} T_i$. Therefore, there exists a partition of unity subordinate to the cover $\{T_i\}_{i\in I}$; i.e, a family of continuous functions $\big\{\psi_i : G \to [0, 1]\big\}_{i\in I}$ with $supp(\psi_i) \subset T_i$ such that for any point $x \in G$, there is a neighborhood of x where all but finite number of functions $\{\psi_i\}_{i\in I}$ are equal to 0, and such that $\sum_{i\in I}\psi_i(x) = 1$. Thus, let $W_1(x) = -\sum_{i\in I}\psi_i(x)\langle\frac{\partial V_i(x)}{\partial x}, F_i(x)\rangle$ which satisfies (III). In a similar manner, for all $Y_l = X_i \cap X_j$ with $(i,j)\in\tilde{I}$ and $l \in L = \{1, 2, ..., M\}$ (where M is the number of members in $\tilde{I}$), there exist open neighborhoods $\mathcal{Y}_l$ whose collection ($\{\mathcal{Y}_l\}_{l\in L}$) is an open cover to the closed set $G' \subset G$. Because G' is a closed subset of G, G' is also compact. So, there exists a partition of unity subordinate to the cover $\{\mathcal{Y}_l\}_{l\in L}$ characterized by $\{\phi_l : \mathcal{Y}_l \to [0,1]\}_{l\in L}$. At this point, it suffices to let $W_2(x) = -\sum_{l\in L}\phi_l(x)\Gamma_l(x)$ where $\Gamma_l(x) = \langle\frac{\partial V_i(x)}{\partial x}, F_j(x)\rangle$ if $x \in Y_l$, $l \in L$. Obviously, $W_2(x)$ satisfies (IV). Finally, we can select the map $W(x) = \max\{W_1(x), W_2(x)\}$, and this completes the proof.

The next proposition provides a Lyapunov-like stability theorem for the class of switched systems under study.

Proposition 10. Let $\{V_i(x)\}_{i\in I}$ be a family of $\mathcal{C}^1$ Lyapunov functions. The switched system $\mathcal{S}$ is asymptotically stable at the origin if the following conditions hold

$$V_i(x) > 0 \qquad \forall x \in X_i \setminus \{0\} \tag{17}$$

$$\langle\frac{\partial V_i(x)}{\partial x}, F_i(x)\rangle < 0 \qquad \forall x \in X_i \setminus \{0\} \tag{18}$$

for all $i \in I$,

$$\langle\frac{\partial V_i(x)}{\partial x}, F_j(x)\rangle < 0 \qquad \forall x \in X_i \cap X_j \setminus \{0\} \tag{19}$$

$$V_i(x) = V_j(x) \qquad \forall x \in X_i \cap X_j \setminus \{0\} \tag{20}$$

for all $(i,j)\in\tilde{I}$

Proof. The proof follows the same lines of Proposition 10 in (Leth and Wisniewski [2012]). It is necessary to show that Proposition 9 holds. From (16),(17),(20), and Proposition 8, we conclude that there exists a continuous, locally Lipschitzean, single-valued, and positive definite function $\Psi(x)$. Subsequently, from (18),(19) and Proposition 9 it follows that there exists a positive definite function $W(x)$ satisfying III and IV.

Given a set of $\mathcal{C}^1$ functions $\{V_i(x)\}_{i\in I}$ and from the definition of partitioning, it follows that for any $x \in G$ and $v \in \mathbb{R}^n$, there is $a > 0$ such that $x + at \in X_j$ for any $t \in [0, a)$. On the other hand, $D_+V_j(x)(v) = \liminf_{h\to 0_+}\frac{V_j(x+hv)-V_j(x)}{h} = \langle\frac{\partial V_j}{\partial x}(x), v\rangle$. Then, from Proposition 9 and (18) it follows that

$$D_+V_i(x)\big(F_i(x)\big) + W(x) \le 0$$

Consequently, for any $u \in \mathcal{F}^c(x)$ and real α_k such that $\sum_{k\in I}\alpha_k = 1$, we arrive at the following justification

$$\begin{aligned} D_+\Psi(x)(u) &= D_+V_i(x)(u) \\ &= D_+V_i(x)\big(\sum_{k\in I}\alpha_k F_k(x)\big) \\ &= \Big\langle\frac{\partial V_i}{\partial x}, \big(\sum_{k\in I}\alpha_k F_k(x)\big)\Big\rangle \\ &= \sum_{k\in I}\alpha_k\langle\frac{\partial V_i}{\partial x}, F_k(x)\rangle \\ &\le -\sum_{k\in I}\alpha_k W(x) \\ &\le -W(x) \end{aligned}$$

in which, we applied (19), condition IV and Proposition 9. Thus, 0 is an asymptotically stable equilibrium.

3.2 Robust Asymptotic Stability of Switched Systems with Polytopic Uncertainty

At this stage, we extend our results to a class of switched systems with polytopic uncertainty $\tilde{\mathcal{S}} = \{G, \mathcal{X}, I, \tilde{\mathscr{F}}\}$ with $\tilde{\mathscr{F}} = \{F_i(x, \theta^i)\}_{i\in I}$ and

$$F_i(x,\theta^i) = \sum_{l=1}^{L_i}\theta_{il}f_{il}(x) \tag{21}$$

where, $f_{il} : U_i \to \mathbb{R}^n, l = 1, 2, ..., L_i$ (U_i is an open neighborhood of X_i) are a family of smooth functions, and θ^i, $i \in I$ are uncertain constant parameter vectors ($\theta^i = [\theta_{i1}, \theta_{i2}, ..., \theta_{iL_i}]^T \in \mathbb{R}^{L_i}$) satisfying

$$\theta^i \in \Theta^i \triangleq \Big\{ \theta^i \in \mathbb{R}^{L_i} \mid \theta_{il} \ge 0,\, l = 1, 2, .., L_i, \text{ and } \sum_{l=1}^{L_i}\theta_{il} = 1\Big\} \tag{22}$$

With the results given in Section 3.1, the following stability theorem for uncertain switched systems with Filippov solutions can be characterized.

Proposition 11. Consider the switched system subject to polytopic uncertainty $\tilde{\mathcal{S}}$. If there exists a family of Lyapunov functions $\{V_i(x)\}_{i\in I}$ satisfying

$$V_i(x) > 0 \qquad \forall x \in X_i \setminus \{0\} \tag{23}$$

$$\langle\frac{\partial V_i(x)}{\partial x}, f_{il}(x)\rangle < 0 \qquad \forall x \in X_i \setminus \{0\} \tag{24}$$

for all $i \in I$ and $l = 1, 2, ..., L_i$,

$$\langle\frac{\partial V_i(x)}{\partial x}, f_{jl}(x)\rangle < 0 \qquad \forall x \in X_i \cap X_j \setminus \{0\} \tag{25}$$

$$V_i(x) = V_j(x) \qquad \forall x \in X_i \cap X_j \setminus \{0\} \tag{26}$$

for all $(i,j)\in\tilde{I}$, and $l = 1, 2, ..., L_j$. Then, the equilibrium point 0 is robustly asymptotically stable.

Proof. This is a direct result of utilizing Proposition 10. (23) and (26) correspond to (17) and (20), respectively. If (24) holds for all $i \in I$ and $l = 1, 2, ..., L_i$, then it follows that for all sets of unknown parameters θ^i satisfying (22)

$$\begin{aligned} \sum_{l=1}^{L_i}\theta_{il}\langle\frac{\partial V_i(x)}{\partial x}, f_{il}(x)\rangle &= \langle\frac{\partial V_i(x)}{\partial x}, \sum_{l=1}^{L_i}\theta_{il}f_{il}(x)\rangle \\ &= \langle\frac{\partial V_i(x)}{\partial x}, F_i(x,\theta^i)\rangle < 0 \end{aligned} \tag{27}$$

which proves that (18) holds. It can be analogously shown that if (25) holds for all $(i,j)\in\tilde{I}$, and $l = 1, 2, ..., L_j$, then it follows that (19) is satisfied for all $(i,j)\in\tilde{I}$. Consequently, by Proposition 10, all Filippov solutions of $\tilde{\mathcal{S}}$ converge to origin asymptotically.

3.3 Sufficient Conditions Based on SOS Decomposition

Henceforth, we focus on a family of polynomial Lyapunov functions. Moreover, we postulate that each F_i, $i \in I$, (correspondingly, f_{il}, $i \in I$, $l = 1, 2, ..., L_i$) is a vector of polynomials in x. The reader should note that Propositions 10 and 11 presents stability conditions for general nonlinear switched systems; nonetheless, the above assumptions on the structure of Lyapunov functions and subsystems should be made so that providing a computational efficient method for stability analysis becomes doable. In order to achieve this goal, we need computational efficient methods to check the positivity of a given polynomial over a specific set. The positivity test can be performed using two main approaches; i.e., the slack variable approach and the SOS approach. The former method has been characterized for positivity analysis of polynomials over hyper-rectangles (Sato [2009]); however, this scheme lacks a computational efficient algorithm. On the other hand, currently there are well-developed computational tools for SOS decomposition e.g. SOSTOOLS (Pranja et al. [2004]).

Let the set

$$\{x \in G \mid \xi_{ik}(x) \geq 0 \quad \text{for} \quad k = 1, 2, \ldots, n_{X_i}\} \tag{28}$$

represent the partition X_i, where $\xi_{ik} \in \mathcal{P}$ and n_{X_i} denotes the number of polynomial inequalities required to completely characterize X_i. The boundary between the partitions is defined as

$$X_i \cap X_j = \{x \in G \mid \gamma_{ij}(x) = 0\} \tag{29}$$

where $\gamma_{ij}(x) \in \mathcal{P}$, with $(i,j) \in \tilde{I}$.

Condition (18) can be guaranteed if there exist polynomial functions $p_{ij}(x) \in \mathcal{P}$ such that

$$V_i(x) + p_{ij}(x)\gamma_{ij}(x) = V_j(x) \quad \text{for all} \quad (i,j) \in \tilde{I} \tag{30}$$

Using the generalized S-procedure (Pólic and Terlaky [2007]), the above discussions, and Proposition 7, it follows that (17) and (18) are satisfied if the following SOS problem is feasible

$$V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \lambda_i \in \mathcal{P}_{sos} \tag{31}$$

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \big\rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{32}$$

for some SOS polynomials $\xi_{ik}(x)$, $w_{ik}(x)$ and positive scalars μ_i, λ_i.

Condition (19) can also be reformulated using Proposition 6 as (see (14) wherein $b_i = 0$, $r_i(x) = r(x) \in \mathcal{P}$, $p = -\big\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \big\rangle$, and noting that $s_0 \in \mathcal{P}_{sos}$)

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, F_j(x) \big\rangle - r(x)\gamma_{ij}(x) - \nu_{ij} \in \mathcal{P}_{sos} \tag{33}$$

Thus far, we have found SOS representations for all conditions in Proposition 10. The following proposition summarizes the above discussions.

Proposition 12. Consider the nonlinear switched system $\mathcal{S}$. If there exist a family of polynomials $\{V_i(x)\}_{i\in I}$ with $V_i(0) = 0$ if $0 \in X_i$, $r_{ij} \in \mathcal{P}$, $p_{ij} \in \mathcal{P}$ with $(i,j) \in \tilde{I}$, $q_{ik} \in \mathcal{P}_{sos}$, $w_{ik} \in \mathcal{P}_{sos}$ with $k = 1, 2, \ldots, n_{X_i}$, and a set of positive scalars λ_i, μ_i with $i \in I$, and ν_{ij} with $(i,j) \in \tilde{I}$, such that

$$V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \lambda_i \in \mathcal{P}_{sos} \tag{34}$$

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \big\rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{35}$$

for all $i \in I$, and

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, F_j(x) \big\rangle - r_{ij}(x)\gamma_{ij}(x) - \nu_{ij} \in \mathcal{P}_{sos} \tag{36}$$

$$V_i(x) + p_{ij}(x)\gamma_{ij}(x) = V_j(x) \tag{37}$$

for all $(i,j) \in \tilde{I}$. Then, the equilibrium 0 is asymptotically stable.

It is worth noting that condition (36) can be further relaxed by just considering those boundaries possessing attractive Filippov solutions (instead of checking (36) for all $(i,j) \in \tilde{I}$). One can infer the existence of an attractive Filippov solution by checking

$$\big\langle \frac{\partial \gamma_{ij}(x)}{\partial x}, F_i(x) \big\rangle \big\langle \frac{\partial \gamma_{ij}(x)}{\partial x}, F_j(x) \big\rangle < 0 \quad \forall x \in X_i \cap X_j \tag{38}$$

or, in terms of an SOS decomposition problem, if the following holds

$$-\big\langle \frac{\partial \gamma_{ij}(x)}{\partial x}, F_i(x) \big\rangle \big\langle \frac{\partial \gamma_{ij}(x)}{\partial x}, F_j(x) \big\rangle - l(x)\gamma_{ij}(x) - \kappa_{ij} \in \mathcal{P}_{sos} \tag{39}$$

for some $l(x) \in \mathcal{P}$ and some positive scalar κ_{ij}.

It should be noted that Proposition 12 only provides sufficient conditions. Indeed, given a nonlinear switched system $\mathcal{S}$, one can search for the corresponding candidate Lyapunov functions via semi-definite programming schemes; if the problem is feasible, then the switched system $\mathcal{S}$ is asymptotically stable.

Based on similar arguments for Proposition 12, we can characterize an SOS representation for conditions in Proposition 11.

Proposition 13. Consider the uncertain switched system $\tilde{\mathcal{S}}$. If there exist a family of polynomials $\{V_i(x)\}_{i\in I}$ with $V_i(0) = 0$ if $0 \in X_i$, $r_{ij} \in \mathcal{P}$, $p_{ij} \in \mathcal{P}$ with $(i,j) \in \tilde{I}$, $q_{ik} \in \mathcal{P}_{sos}$, $w_{ik} \in \mathcal{P}_{sos}$ with $k = 1, 2, \ldots, n_{X_i}$, and a set of positive scalars λ_{il}, μ_{il} with $i \in I$ $(l = 1, 2, ..., L_i)$, and ν_{ijl} with $(i,j) \in \tilde{I}$ $(l = 1, 2, ..., L_j)$, such that

$$V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \lambda_{il} \in \mathcal{P}_{sos} \tag{40}$$

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, f_{il}(x) \big\rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_{il} \in \mathcal{P}_{sos} \tag{41}$$

holds for all for all $i \in I$ and $l = 1, 2, ..., L_i$,

$$-\big\langle \frac{\partial V_i(x)}{\partial x}, f_{jl}(x) \big\rangle - r_{ij}(x)\gamma_{ij}(x) - \nu_{ijl} \in \mathcal{P}_{sos} \tag{42}$$

and (37) holds for all for all $(i,j) \in \tilde{I}$ and $l = 1, 2, ..., L_j$. Then, the origin is robustly asymptotically stable.

4. SIMULATION EXAMPLE

Consider the following switched system described by

$$\dot{x} \in \mathcal{G}(x) \tag{43}$$

$$\dot{x} \in \mathcal{G}^c(x) \tag{44}$$

with $\mathcal{G} : \mathbb{R}^2 \to 2^{\mathbb{R}^2}; x \mapsto \{v \in \mathbb{R}^2 \mid v = g_i(x) \text{ if } x \in X_i\}$ $(\mathcal{G}^c = co(\mathcal{G}))$, wherein $i \in I = \{1, 2\}$, the partitions defined as

$$X_1 = \{x \in \mathbb{R}^2 \mid x_2^2 - x_1^3 > 0\} \tag{45}$$

$$X_2 = \{x \in \mathbb{R}^2 \mid x_2^2 - x_1^3 < 0\} \tag{46}$$

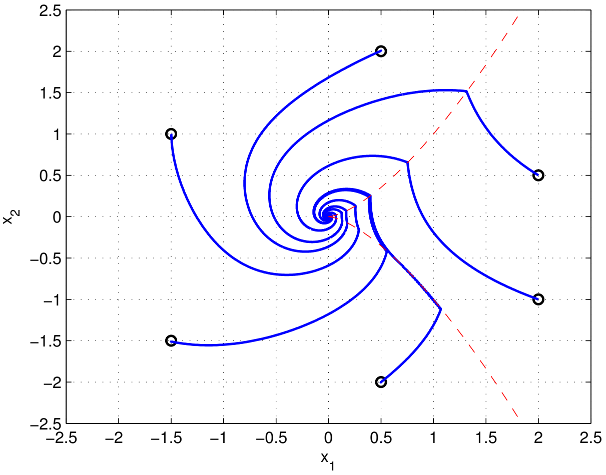

Fig. 2. Trajectories of the example switched system. Note that there exists an attractive Filippov solution.

and the subsystems given by

$$g_1(x) = \begin{bmatrix} -2x_1 - x_1^3 - 5x_2 - x_2^3 \\ 6x_1 + x_1^3 - 3x_2 - x_2^3 \end{bmatrix} \tag{47}$$

$$g_2(x) = \begin{bmatrix} x_2 + x_1^2 - x_1^3 \\ 4x_1 + 2x_2 \end{bmatrix} \tag{48}$$

It is worth noting that although subsystem $g_1(x)$ is asymptotically stable at the origin, $g_2(x)$ is unstable; therefore, a unified Lyapunov function may not exist. However, the simulations show that the overall switched system is stable at the origin (see Fig. 2). The stability of the above system can be verified using the results obtained in Proposition 12. Using SOSTOOLS v. 2.03, two candidate Lyapunov functions of degree six were found

$$\begin{aligned} V_1(x) = 24.445x_1^6 - 9.763x1^4 + 1.948x_1^3x_2^2 \\ + 3.203x_2^4 + 1.008x_2^6 \end{aligned} \tag{49}$$

$$V_2(x) = V_1(x) + x_2^2 - x_1^3 \tag{50}$$

This is consistent with the results presented in this paper. We remark that for polynomials of lesser degree say 4 the search for a candidate Lyapunov function was infeasible. Note that (50) ensures that condition (37) is satisfied. In this simulation, the corresponding values for scalars $\{\lambda_i\}_{i\in I}$, $\{\mu_i\}_{i\in I}$, and $\{\nu_{ij}\}_{(i,j)\in\tilde{I}}$ were set to 0.1. The SOS polynomials $\{w_{i1}\}_{i\in I}$ and $\{\xi_{i1}\}_{i\in I}$ were set to $(x_2^2 - x_1^3)^2$.

5. CONCLUSION

In this paper, a Lyapunov-like stability theorem for nonlinear switched systems with partitioned state-space and state-dependent switching is brought forward. This result has been exploited to formulate conditions on robust asymptotic stability of switched systems with polytopic uncertainty. Since the analysis is based on the theory of differential inclusions, the proposed stability analysis scheme includes Filippov solutions. Furthermore, in order to provide a computationally efficient method to implement the suggested stability theorem, the results are reformulated using SOS decomposition techniques which can be realized based on semi-definite programming tools.

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