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Robust Stability and H_∞ Control of Uncertain Piecewise Linear Switched Systems with Filippov Solutions

Mohamadreza Ahmadi[1], *Member, IEEE*, Hamed Mojallali[1], Rafael Wisniewski[2] and Roozbeh Izadi-Zamanabadi[2]

Abstract— **This paper addresses the robust stability and control problem of uncertain piecewise linear switched systems where, instead of the conventional Carathéodory solutions, we allow for Filippov solutions. In other words, in contrast to the previous studies, solutions with infinite switching in finite time along the facets and on faces of arbitrary dimensions are also taken into account. Firstly, based on earlier results, the stability problem of piecewise linear systems with Filippov solutions is translated into a number of linear matrix inequality feasibility tests. Subsequently, a set of matrix inequalities are brought forward, which determines the asymptotic stability of the Filippov solutions of a given uncertain piecewise linear system. Afterwards, bilinear matrix inequality conditions for synthesizing a robust controller with a guaranteed H_∞ performance are formulated. Finally, a V-K iteration algorithm is proposed to surmount the aforementioned matrix inequality conditions.**

I. INTRODUCTION

Piecewise linear (PWL) systems are an important class of hybrid systems, which has received tremendous attention in open literature [1]-[10]. By a PWL system, we understand a family of linear systems defined on polyhedral sets such that the dynamics inside a polytope is governed by a linear dynamic equation. The union of these polyhedral sets forms the state-space. We say that a "switch" has occurred whenever the trajectories passes to an adjacent polytope.

The stability analysis of PWL systems is a complicated assignment. It is well known that even if all the subsystems are stable, the overall system may possess divergent trajectories [10]. Furthermore, the behavior of solutions along the boundary of polytopes (facets) may engender unstable trajectories where transitions are, generally speaking, multi-valued. That is, a PWL system with stable Carathéodory solution may possess divergent Filippov solutions such that the overall system is unstable (see Example 5 in [7]). Hence, the stability of the Carathédory solutions does not imply the stability of the overall PWL system.

The stability problem of PWL systems has been addressed by a number of researchers [3]-[7]. An efficacious contribution was made by Johansson and Rantzer in [3]. The authors proposed a number of LMI feasibility tests to investigate the exponential stability of a given PWL system by introducing the concept of piecewise quadratic Lyapunov functions. Following the same trend, Chan et al. [5] extended the results to the case of uncertain PWL systems. The authors also brought forward a H_∞ controller synthesis scheme for uncertain PWL systems based on a set of LMI conditions.

However, the solutions considered implicitly in both contributions are defined in the sense of Carathéodory. This means that a solution of a PWL system is the concatenation of classical solutions on the facets of polyhedral sets. In other words, sliding phenomena or solutions with infinite switching in finite time are inevitably eliminated from the analyses. In this study, in lieu of the Carathéodory solutions, the more universal Filippov solutions [11] are considered. Our approach has its roots in [7], wherein the authors applied the theory of differential inclusions to derive stability theorems for switched systems with Filippov solutions. The results reported in this paper are formulized as a set of LMI or bilinear matrix inequality (BMI) conditions which can be formulated into a semi-definite programming problem.

The framework of this paper is organized as follows. The subsequent section discusses the robust stability results. In Section III, a stabilizing state-feedback controller for uncertain PWL systems is formulated. The H_∞ Controller synthesis methodology and a V-K iteration algorithm to deal with the BMI conditions are described in Section IV. The accuracy of the proposed method is evaluated by a simulation example in Section V. The paper ends with conclusions in Section VI.

II. ROBUST STABILITY ANALYSIS

A. PWL Systems with Filippov Solutions

We will study a class of PWL systems with Filippov solutions $\mathcal{S} = \{\mathcal{X}, \mathcal{U}, \mathcal{V}, X, I, F, G\}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ is a polyhedral set representing the state space, $\mathcal{X} = \{X_i\}_{i \in I}$ is the set containing the polytopes in $\mathcal{X}$ with index set $I = \{1, 2, \ldots, n_X\}$ (note that $\bigcup_{i \in I} X_i = \mathcal{X}$). Each polytope X_i is characterized by the set $\{x \in \mathcal{X} \mid E_i x \succcurlyeq 0\}$ where the notation $\succcurlyeq$ signifies the component-wise inequality. $\mathcal{U}$ is the control space and $\mathcal{V}$ is the disturbance space, which are both subsets of Euclidean spaces. In addition, $v \in \mathcal{L}_2[0, \infty)$. $F = \{f_i\}_{i \in I}$ and $G = \{g_i\}_{i \in I}$ are families of linear functions associated with the system states x and outputs y. Each f_i consists of six elements $(A_i, B_i, D_i; \Delta A_i, \Delta B_i, \Delta D_i)$ and each g_i is composed of four elements $(C_i, G_i; \Delta C_i, \Delta G_i)$. Furthermore, $f_i : Y_i \times \mathcal{U} \times \mathcal{V} \to \mathbb{R}^n; (x, u, v) \mapsto \{z \in \mathbb{R}^n \mid$

[1]M. Ahmadi, and H. Mojallai are with the Department of Electrical Engineering, University of Guilan, Rasht, Iran, P.O. Box: 41635-3756, Postal Code: 41996-13769 mrezaahmadi at ieee.org; mojallali at guilan.ac.ir

[2] R. Wisniewski, and R. Izadi-Zamanabadi are with the Section of Automation & Control, Aalborg University, 9220 Aalborg East, Denmark raf at es.aau.dk; riz at es.aau.dk

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$z = (A_i + \Delta A_i)x + (B_i + \Delta B_i)u + (D_i + \Delta D_i)v\big\}$ and $g_i : Y_i \times \mathcal{U} \to \mathbb{R}^m; (x,u) \mapsto \big\{z \in \mathbb{R}^m \mid z = (C_i + \Delta C_i)x + (G_i + \Delta G_i)u\big\}$ where Y_i is an open neighborhood of X_i. The set of matrices $(A_i, B_i, C_i, D_i, G_i)$ are defined over the polytope X_i and $(\Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i, \Delta G_i)$ encompass the corresponding uncertainty terms. The dynamics of the system can be described by

$$\dot{x}(t) \in co\Big(\mathcal{F}\big(x(t), u(t), v(t)\big)\Big) \quad (1)$$

$$y(t) \in \mathcal{G}\big(x(t), u(t)\big) \quad (2)$$

where, $co(\cdot)$ denotes the convex hull, the set valued maps $\mathcal{F}$ and $\mathcal{G}$ are defined as

$$\begin{aligned} \mathcal{F} \; &: \; \mathcal{X} \times \mathcal{U} \times \mathcal{V} \to 2^{\mathcal{X}} \\ &; \; (x,u,v) \mapsto \{z \in \mathbb{R}^n \mid z = f_i(x,u,v) \text{ if } x \in X_i\} \end{aligned} \quad (3)$$

$$\begin{aligned} \mathcal{G} \; &: \; \mathcal{X} \times \mathcal{U} \to 2^{\mathbb{R}^m} \\ &; \; (x,u) \mapsto \{z \in \mathbb{R}^m \mid z = g_i(x,u) \text{ if } x \in X_i\} \end{aligned} \quad (4)$$

where the notation 2^A means the power set or the set of all subsets of A. Denote by $\tilde{I} = \{(i,j) \in I^2 \mid X_i \cap X_j \neq \emptyset, \; i \neq j\}$ the set of index pairs which determines the polytopes with non-empty intersections. We now assume that each polytope is the intersection of a finite set of supporting half spaces. By N_{ij} denote the normal vector pertained to the hyperplane supporting both X_i and X_j. Consequently, each boundary can be characterized as

$$X_i \cap X_j = \{x \in \mathcal{X} \mid N_{ij}^T x \approx 0, \; H_{ij}x \succcurlyeq 0, \; (i,j) \in \tilde{I}\} \quad (5)$$

where $\approx$ represent the component-wise equality and the inequality $H_{ij}x \succcurlyeq 0$ confines the hyperplane to the interested region. Throughout the paper, the matrix inequalities should be understood in the sense of positive definiteness; i.e., $A > B$ $(A \geq B)$ means $A - B$ is positive definite (semi-definite). In case of matrix inequalities, I denotes the unity matrix (the size of I can be inferred from the context) and should be distinguished from the index set I. In matrices, $\star$ in place of a matrix entry a_{mn} means that $a_{mn} = a_{nm}^T$.

A Filippov solution to (1) is an absolutely continuous function $[0,T) \to \mathcal{X}; t \mapsto \phi(t)$ $(T > 0)$ which solves the following Cauchy problem

$$\dot{\phi}(t) \in co\Big(\mathcal{F}\big(\phi(t), u(t), v(t)\big)\Big) \quad \text{a.e.,} \quad \phi(0) = \phi_0 \quad (6)$$

In the sequel, we assume that at any interior point $x \in \mathcal{X}$ there exists a Filippov solution to system (1). This can be evidenced by Proposition 5 in [7]. For more information pertaining to the solutions and their existence or uniqueness properties, the interested reader is referred to the expository review [12] and the book [11].

In [7], Leth and Wisniewski proposed a stability theorem for switched systems with Filippov solutions which is reformulated for PWL systems in the next proposition.

Proposition 2.1: Consider the following autonomous PWL system

$$\dot{x} \in co\big(\mathcal{F}(x)\big) \quad (7)$$

with $\Delta A_i \approx 0$. If there exists quadratic forms $\Phi_i(x) = x^T Q_i x$, $\Psi_i(x) = x^T(A_i^T Q_i + Q_i A_i)x$ and $\Psi_{ij}(x) = x^T(A_j^T Q_i + Q_i A_j)x$ satisfying

$$\Phi_i(x) > 0 \quad \text{for all} \quad x \in X_i \setminus \{0\} \quad (8)$$

$$\Psi_i(x) < 0 \quad \text{for all} \quad x \in X_i \setminus \{0\} \quad (9)$$

for all $i \in I$, and

$$\Psi_{ij}(x) < 0 \quad \text{for all} \quad x \in X_i \cap X_j \setminus \{0\} \quad (10)$$

$$\Phi_i(x) = \Phi_j(x) \quad \text{for all} \quad x \in X_i \cap X_j \quad (11)$$

for all $(i,j) \in \tilde{I}$. Then, the the equilibrium point 0 of (7) is asymptotically stable.

Remark 2.1: The inclusions $x \in X_i \setminus \{0\}$ and $x \in X_i \cap X_j$ are analogous to $\{x \in \mathcal{X} \mid E_i x \succ 0\}$ and (5), respectively. It is worth noting that Conditions (8)-(9) are concerned with the positivity of a quadratic form over a polytope; whereas, (10) is about positivity over a hyperplane. Condition (11) asserts that the candidate Lyapunov functions should be continuous (along the facets). A well known LMI formulation of conditions (8), (9) and (11) was proposed in [3] which is described next. Let us construct a set of matrices F_i, $i \in I$ such that $F_i x = F_j x$ for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$. Then, it follows that the piecewise linear candidate Lyapunov functions can be formulated as

$$V(x) = x^T F_i^T M F_i x = x^T Q_i x \quad \text{if} \quad x \in X_i \quad (12)$$

where, the free parameters of Lyapunov functions are concentrated in the symmetric matrix M. In the following proposition we generalize the results proposed by Johansson and Rantzer [3] to PWL systems with the more general Filippov solutions.

Proposition 2.2: Consider the PWL system (7) with Fillipov solutions, and the family of piecewise quadratic Lyapunov functions $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$. If there exist a set of symmetric matrices Q_i, three sets of symmetric matrices U_i, S_i, T_{ij} with non-negative entries, and matrices W_{ij} of appropriate dimensions with $i \in I$ and $(i,j) \in \tilde{I}$, such that the following LMI problem is feasible

$$Q_i - E_i^T S_i E_i > 0 \quad (13)$$

$$A_i^T Q_i + Q_i A_i + E_i^T U_i E_i < 0 \quad (14)$$

for all $i \in I$, and

$$A_j^T Q_i + Q_i A_j + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} < 0 \quad (15)$$

for all $(i,j) \in \tilde{I}$. Then, the equilibrium point 0 of (7) is asymptotically stable.

Proof: Matrix inequalities (13) and (14) are the same as Equation (11) in Theorem 1 in [3] which satisfy (8)-(9). The continuity of the Lyapunov functions is also ensured from the assumption that $V_i(x) = x^T Q_i x = x^T F_i^T M F_i x$, $i \in I$ since $F_i x = F_j x$, for all $x \in X_i \cap X_j$ and $(i,j) \in \tilde{I}$.

(10) is equivalent to $x^T(A_j^TQ_i + Q_iA_j)x < 0$ for $\{x \in \mathcal{X} \mid N_{ij}^Tx \approx 0,\ H_ijx \succ 0\}$. Applying the S-procedure and Finsler's lemma [13], we obtain (15) for a set of matrices T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries and W_{ij}, $(i,j) \in \tilde{I}$ with appropriate dimensions. ■

We remark that algorithms for constructing matrices E_i and F_i, $i \in I$, are described in [8].

Remark 2.2: A similar LMI formulation to (11) can be found in [8]; whereas, our analysis, in this paper, is established upon the stability theorem delineated in Proposition 10 in [7] which considered the Filippov Solutions.

B. Uncertain PWL Systems with Filippov Solutions

Henceforth, we will focus on the family of uncertain PWL systems given by (1). In order to derive the stability and control results, we assume that the upper bound of uncertainties are known apriori; i.e.,

$$\begin{aligned} {\Delta A_i}^T\Delta A_i &\leq \mathcal{A}_i^T\mathcal{A}_i \\ {\Delta B_i}^T\Delta B_i &\leq \mathcal{B}_i^T\mathcal{B}_i \\ {\Delta C_i}^T\Delta C_i &\leq \mathcal{C}_i^T\mathcal{C}_i \\ {\Delta D_i}^T\Delta D_i &\leq \mathcal{D}_i^T\mathcal{D}_i \\ {\Delta G_i}^T\Delta G_i &\leq \mathcal{G}_i^T\mathcal{G}_i \end{aligned} \tag{16}$$

in which, $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i, \mathcal{G}_i)$ are any set of constant matrices with the same dimension as $(A_i, B_i, C_i, D_i, G_i)$ satisfying (16).

Proposition 2.3: Consider the uncertain PWL system (7). If there exist small positive constants ϵ_i, $i \in I$, ϵ_{ij}, $(i,j) \in \tilde{I}$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$, with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$, of appropriate dimensions, such that

$$Q_i - E_i^TS_iE_i > 0 \tag{17}$$

$$\begin{bmatrix} \Xi_i & Q_i \\ \star & -\epsilon_i \mathrm{I} \end{bmatrix} < 0 \tag{18}$$

for all $i \in I$, and

$$\begin{bmatrix} \Xi_{ij} & Q_i \\ \star & -\epsilon_{ij} \mathrm{I} \end{bmatrix} < 0 \tag{19}$$

for all $(i,j) \in \tilde{I}$, where $\Xi_i = A_i^TQ_i + Q_iA_i + E_i^TU_iE_i + \epsilon_i\mathcal{A}_i^T\mathcal{A}_i$ and $\Xi_{ij} = A_j^TQ_i + Q_iA_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} + \epsilon_{ij}\mathcal{A}_j^T\mathcal{A}_j$. Then, every Filippov solution of the autonomous uncertain system (7) converges to the origin asymptotically.

Proof: Condition (17) is equivalent to (13). We need to show that (18) and (19) correspond to (14) and (15), respectively. Substituting the uncertain vector $\bar{A}_i = A_i + \Delta A_i$ in (15) yields $(A_j + \Delta A_j)^TQ_i + Q_i(A_j + \Delta A_j) + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} < 0$ which with little manipulation leads to $A_j^TQ_i + Q_iA_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} + \Delta A_j^TQ_i + Q_i\Delta A_j \leq A_j^TQ_i + Q_iA_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} + \frac{1}{\epsilon_{ij}}Q_iQ_i + \epsilon_{ij}\mathcal{A}_j^T\mathcal{A}_j$. Using Shur complement theorem, we arrive at (19). The equivalency of (18) to (14) can also be proved in a similar manner. ■

Remark 2.3: Notice that if the conditions (17)–(19) hold, then (7) is also asymptotically stable for any ΔA_i satisfying (16).

III. Stabilizing State Feedback Controller Design

We are interested in designing a switching controller

$$\begin{gathered} u \in \mathcal{K}(x) \\ \mathcal{K} : \mathcal{X} \to 2^{\mathcal{U}}; x \mapsto \left\{z \in \mathcal{U} \mid z = K_ix \text{ if } x \in X_i\right\} \end{gathered} \tag{20}$$

for system (1) such that all Filippov solutions of (1) ($\phi(t)$) satisfy $\lim_{t\to\infty}\phi(t) = 0$. Considering a controller with the structure given by (20), the controlled system with $v \approx 0$ reduces to (7) with $\mathcal{F}$ supplanted by $\acute{\mathcal{F}} : \mathcal{X} \to 2^{\mathcal{X}}; x \mapsto \{z \in \mathcal{X} \mid z = A_{ci}x \text{ if } x \in X_i\}$, wherein $A_{ci} = A_i + \Delta A_i + (B_i + \Delta B_i)K_i$.

Lemma 3.1: The controlled switched system as defined above is asymptotically stable at the origin provided that there exist small positive constants ϵ_i, $i \in I$, ϵ_{ij}, $(i,j) \in \tilde{I}$, matrices K_i, $i \in I$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries, matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^TS_iE_i > 0 \tag{21}$$

$$\begin{bmatrix} \Xi_i & Q_i & K_i^TB_i^T & K_i^T\mathcal{B}_i^T \\ \star & \frac{-\epsilon_i}{3+\epsilon_i^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_i}{1+\epsilon_i^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_i}\mathrm{I} \end{bmatrix} < 0 \tag{22}$$

for all $i \in I$, and

$$\begin{bmatrix} \Xi_{ij} & Q_i & K_j^TB_j^T & K_j^T\mathcal{B}_j^T \\ \star & \frac{-\epsilon_{ij}}{3+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 \\ \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_{ij}}\mathrm{I} \end{bmatrix} < 0 \tag{23}$$

for all $(i,j) \in \tilde{I}$.

Proof: We need to demonstrate that (22) and (23) correspond to (14) and (15), respectively. Substituting A_{ci} in (15) yields $A_j^TQ_i + Q_iA_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} + \Delta A_j^TQ_i + Q_i\Delta A_j + K_j^TB_j^TQ_i + Q_iB_jK_j + K_j^T\Delta B_j^TQ_i + Q_i\Delta B_jK_j \leq A_j^TQ_i + Q_iA_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^TT_{ij}H_{ij} + \frac{1}{\epsilon_{ij}}Q_iQ_i + \epsilon_{ij}\Delta A_j^T\Delta A_j + \frac{1}{\epsilon_{ij}}Q_iQ_i + \epsilon_{ij}K_j^T\Delta B_j^T\Delta B_jK_j + K_j^TB_j^TQ_i + Q_iB_jK_j \leq \Xi_{ij} + (\epsilon_{ij} + \frac{3}{\epsilon_{ij}})Q_iQ_i + (\epsilon_{ij} + \frac{1}{\epsilon_{ij}})K_j^TB_j^TB_jK_j + \epsilon_{ij}K_j^T\mathcal{B}_j^T\mathcal{B}_jK_j$. Utilizing Shur complement theorem, we derive (23). Derivation of (22) can be done similarly. ■

Remark 3.1: The conditions derived in Lemma 3.1 are BMIs [14] in the variables Q_i and K_i.

IV. Robust Controller Synthesis with H_∞ Performance

In this section, we propose a set of conditions to design a stabilizing switching controller of the form (20) with a guaranteed H_∞ performance. That is, a controller such that, in addition to asymptotic stability, ensures that the induced

$\mathcal{L}_2$-norm of the operator from $v(t)$ to the controller output $y(t)$ is less than a constant $\eta > 0$ under zero initial conditions ($x(0) = 0$); in other words,

$$\frac{1}{2}\bigg(\int_0^{\infty} y^T(\tau)y(\tau)\mathrm{d}\tau\bigg)^{\frac{1}{2}} \leq \frac{\eta}{2}\bigg(\int_0^{\infty} v^T(\tau)v(\tau)\mathrm{d}\tau\bigg)^{\frac{1}{2}} \tag{24}$$

given any non-zero $v \in \mathcal{L}_2[0,\infty)$.

If we apply the switching controller (20) to (1)-(2), we arrive at the following controlled system with outputs

$$\begin{aligned} \dot{x}(t) &\in co\Big(\tilde{\mathcal{F}}\big(x(t), v(t)\big)\Big) \\ y(t) &\in \tilde{\mathcal{G}}\big(x(t)\big) \end{aligned} \tag{25}$$

where, $\tilde{\mathcal{F}} : \mathcal{X} \times \mathcal{V} \to 2^{\mathcal{X}}; (x, v) \mapsto \{z \in \mathbb{R}^n \mid z = A_{ci}x + D_{ci}v$ if $x \in X_i\}$ and $\mathcal{G} : \mathcal{X} \to 2^{\mathbb{R}^m}; x \mapsto \{z \in \mathbb{R}^m \mid z = C_{ci}(x)$ if $x \in X_i\}$ with

$$\begin{aligned} A_{ci} &= A_i + \Delta A_i + (B_i + \Delta B_i)K_i \\ D_{ci} &= D_i + \Delta D_i \\ C_{ci} &= C_i + \Delta C_i + (G_i + \Delta G_i)K_i \end{aligned} \tag{26}$$

Proposition 4.1: System (25) is asymptotically stable at the origin with disturbance attenuation η as defined in (24), if there exist a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in \tilde{I}$ with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^T S_i E_i > 0 \tag{27}$$

$$A_{ci}^T Q_i + Q_i A_{ci} + E_i^T U_i E_i + \eta^{-2} Q_i D_{ci} D_{ci}^T Q_i + C_{ci}^T C_{ci} < 0 \tag{28}$$

for all $i \in I$, and

$$\begin{aligned} A_{cj}^T Q_i &+ Q_i A_{cj} + W_{ij} N_{ij}^T + N_{ij} W_{ij}^T + H_{ij}^T T_{ij} H_{ij} \\ &+ \eta^{-2} Q_i D_{cj} D_{cj}^T Q_i + C_{cj}^T C_{cj} < 0 \end{aligned} \tag{29}$$

for all $(i,j) \in \tilde{I}$.

Proof: The proof is omitted here due to lack of space. ■

Lemma 4.2: Given a constant $\eta > 0$, the closed loop control system (25) is asymptotically stable at the origin with disturbance attenuation η, if there exist constants $\epsilon_{ij} > 0$, $(i,j) \in \tilde{I}$, $\epsilon_i > 0$, $i \in I$, matrices K_i, $i \in I$, a set of symmetric matrices Q_i, $i \in I$, three sets of symmetric matrices U_i, S_i, $i \in I$, T_{ij}, $(i,j) \in I$ with non-negative entries, and matrices W_{ij}, $(i,j) \in \tilde{I}$ of appropriate dimensions such that

$$Q_i - E_i^T S_i E_i > 0 \tag{30}$$

$$\Lambda_i < 0 \tag{31}$$

for all $i \in I$, and

$$\Lambda_{ij} < 0 \tag{32}$$

for all $(i,j) \in \tilde{I}$, where

$$\Lambda_i = \begin{bmatrix} \Pi_i & Q_i & K_i^T B_i^T & K_i^T \mathcal{B}_i^T & K_i^T G_i^T & K_i^T \mathcal{G}_i^T \\ \star & -\Theta_i^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \frac{-\epsilon_i}{1+\epsilon_i^2}\mathrm{I} & 0 & 0 & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_i}\mathrm{I} & 0 & 0 \\ \star & \star & \star & \star & \frac{-\epsilon_i}{2+\epsilon_i+\epsilon_i^2}\mathrm{I} & 0 \\ \star & \star & \star & \star & \star & \frac{-\epsilon_i}{1+\epsilon_i+2\epsilon_i^2}\mathrm{I} \end{bmatrix}$$

$$\Lambda_{ij} = \begin{bmatrix} \Pi_{ij} & Q_i & K_j^T B_j^T & K_j^T \mathcal{B}_j^T & K_j^T G_j^T & K_j^T \mathcal{G}_j^T \\ \star & -\Theta_{ij}^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}^2}\mathrm{I} & 0 & 0 & 0 \\ \star & \star & \star & \frac{-1}{\epsilon_{ij}}\mathrm{I} & 0 & 0 \\ \star & \star & \star & \star & \frac{-\epsilon_{ij}}{2+\epsilon_{ij}+\epsilon_{ij}^2}\mathrm{I} & 0 \\ \star & \star & \star & \star & \star & \frac{-\epsilon_{ij}}{1+\epsilon_{ij}+2\epsilon_{ij}^2}\mathrm{I} \end{bmatrix}$$

with $\Pi_i = \Xi_i + (1 + \frac{3}{\epsilon_i})C_i^T C_i + (1 + 3\epsilon_i)\mathcal{C}_i^T \mathcal{C}_i$, $\Pi_{ij} = \Xi_{ij} + (1 + \frac{3}{\epsilon_{ij}})C_j^T C_j + (1 + 3\epsilon_{ij})\mathcal{C}_j^T \mathcal{C}_j$, $\Theta_i = (\epsilon_i + \frac{3}{\epsilon_i})I + \eta^{-2}(1 + \frac{1}{\epsilon_i})D_i D_i^T + \eta^{-2}(1 + \epsilon_i)\mathcal{D}_i \mathcal{D}_i^T$, and $\Theta_{ij} = (\epsilon_{ij} + \frac{3}{\epsilon_{ij}})I + \eta^{-2}(1 + \frac{1}{\epsilon_{ij}})D_j D_j^T + \eta^{-2}(1 + \epsilon_{ij})\mathcal{D}_j \mathcal{D}_j^T$.

Proof: We need to apply Proposition 4.1. Inequality (30) corresponds to (27). Substituting (26) in (29), the left-hand side of (29) is simplified as $LHS = \big(A_j + \Delta A_j + (B_j + \Delta B_j)K_j\big)^T Q_i + Q_i\big(A_j + \Delta A_j + (B_j + \Delta B_j)K_j\big) + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + \eta^{-2} Q_i (D_j + \Delta D_j)(D_j + \Delta D_j)^T Q_i + \big(C_j + \Delta C_j + (G_i + \Delta G_i)K_j\big)^T \big(C_j + \Delta C_j + (G_i + \Delta G_i)K_j\big)$ $\leq A_j^T Q_i + Q_i A_j + W_{ij}N_{ij}^T + N_{ij}W_{ij}^T + H_{ij}^T T_{ij} H_{ij} + K_j^T B_j^T Q_i + Q_i B_j K_j + \frac{2}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij} \mathcal{A}_j^T \mathcal{A}_j + \epsilon_{ij} K_j^T \mathcal{B}_j^T \mathcal{B}_j K_j + \eta^{-2} Q_i \big((1 + \frac{1}{\epsilon_{ij}}) D_j D_j^T + (1 + \epsilon_{ij}) \mathcal{D}_j \mathcal{D}_j^T\big) Q_i + (1 + \epsilon_{ij}) C_j^T C_j + (1 + \epsilon_{ij}) \mathcal{C}_j^T \mathcal{C}_j + \frac{1}{\epsilon_{ij}} C_j^T C_j + \epsilon_{ij} K_j^T G_j^T G_j K_j + \frac{1}{\epsilon_{ij}} C_j^T C_j + \epsilon_{ij} K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j + \epsilon_{ij} \mathcal{C}_j^T \mathcal{C}_j + \frac{1}{\epsilon_{ij}} K_j^T G_j^T G_j K_j + \epsilon_{ij} \mathcal{C}_j^T \mathcal{C}_j + \frac{1}{\epsilon_{ij}} K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j + K_j^T \big((1 + \frac{1}{\epsilon_{ij}}) G_j^T G_j + (1 + \epsilon_{ij}) \mathcal{G}_j^T \mathcal{G}_j\big) K_j$.

With some calculation, it can be verified that $LHS \leq \Pi_{ij} + Q_i\big(\frac{2}{\epsilon_{ij}} I + \eta^{-2}(1 + \frac{1}{\epsilon_{ij}}) D_j^T D_j + \eta^{-2}(1 + \epsilon_{ij}) \mathcal{D}_j \mathcal{D}_j^T\big) Q_i + \epsilon_{ij} K_j^T \mathcal{B}_j^T \mathcal{B}_j K_j + (\frac{2+\epsilon_{ij}+\epsilon_{ij}^2}{\epsilon_{ij}}) K_j^T G_j^T G_j K_j + (\frac{1+\epsilon_{ij}+2\epsilon_{ij}^2}{\epsilon_{ij}} K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j) + \frac{1}{\epsilon_{ij}} K_j^T B_j^T B_j K_j + \epsilon_{ij} Q_i Q_i + \frac{1}{\epsilon_{ij}} Q_i Q_i + \epsilon_{ij} K_j^T B_j^T B_j K_j$ which is equivalent to $LHS \leq \Pi_{ij} + Q_i \Theta_{ij} Q_i + (\frac{1+\epsilon_{ij}^2}{\epsilon_{ij}}) K_j^T B_j^T B_j K_j + \epsilon_{ij} K_j^T \mathcal{B}_j^T \mathcal{B}_j K_j + (\frac{2+\epsilon_{ij}+\epsilon_{ij}^2}{\epsilon_{ij}}) K_j^T G_j^T G_j K_j + (\frac{1+\epsilon_{ij}+2\epsilon_{ij}^2}{\epsilon_{ij}}) K_j^T \mathcal{G}_j^T \mathcal{G}_j K_j$.

Utilizing Shur complement theorem, (32) can be obtained. Thus, if (32) is feasible, then (29) is satisfied. Analogously, it can be proved that (31) is consistent with (28). ■

It is worth noting that conditions given in Lemma 4.2 are BMIs in matrix variables K_i and Q_i. In order to deal with the BMI conditions encountered in Lemmas 3.1 and 4.2, the following $V - K$ iteration algorithm is suggested:

- *Initialization*: Select a set of controller gains based on pole placement method or any other controller design scheme to predetermine a set of initial controller gains.

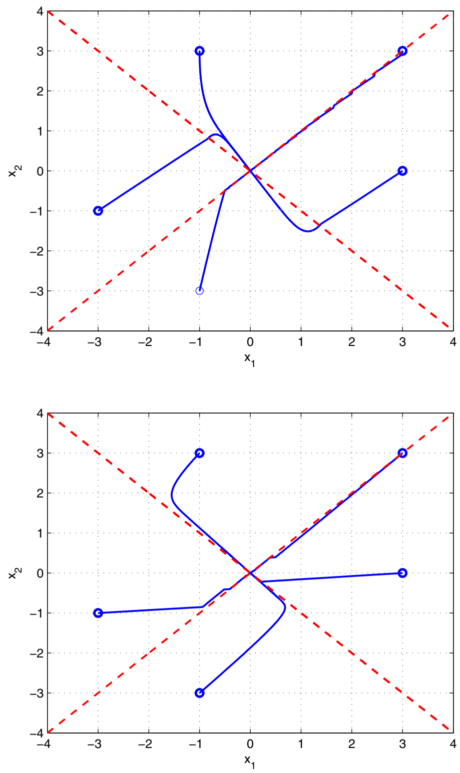

Fig. 1. The responses of the closed loop control system: the stable controller synthesis (top) and the H_∞ controller synthesis (bottom). The dashed lines illustrate the facets.

- *Step V*: Given the set of fixed controller gains K_i, $i \in I$, solve the following optimization problem

$$\min_{Q_i, S_i, U_i, T_{ij}} \gamma_i$$
$$\text{subject to (30)}, \Lambda_i - \gamma_i \mathrm{I} < 0, \text{and } \Lambda_{ij} - \gamma_i \mathrm{I} < 0$$

  for a set of matrices Q_i, $i \in I$.
- *Step K*: Given the set of fixed controller gains Q_i, $i \in I$, solve the following optimization problem

$$\min_{K_i, S_i, U_i, T_{ij}} \gamma_i$$
$$\text{subject to (30)}, \Lambda_i - \gamma_i \mathrm{I} < 0, \text{and } \Lambda_{ij} - \gamma_i \mathrm{I} < 0$$

  for a set of matrices K_i, $i \in I$.

The algorithm continues till $\gamma_i < 0$, $i \in I$.

Remark 4.1: Generalization of the results presented in this paper to the case of piecewise affine (PWA) dynamics is straightforward. This can be simply realized by augmenting the corresponding system matrices as demonstrated in [8].

V. Simulation Example

In this section, we demonstrate the performance of the proposed approach using a numerical example. For the sake of comparison, the example used in [5] is selected; but, instead of Carathéodory solutions, Filippov solutions are investigated. Therefore, the system structure has to be modified as delineated next. Consider an uncertain PWL system described by (25) and (26) with $I = \{1, 2, 3, 4\}$ and the state-space is a polyhedral set divided into four polytopes. The associated system matrices are

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

The uncertainty bounds are characterized as

$$\mathcal{A}_1 = \mathcal{A}_3 = \begin{bmatrix} 0 & 0.02 \\ -0.01 & 0 \end{bmatrix}, \mathcal{A}_2 = \mathcal{A}_4 = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.02 \end{bmatrix}$$

$$\mathcal{B}_1 = \mathcal{B}_3 = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \mathcal{B}_2 = \mathcal{B}_4 = \begin{bmatrix} 0.02 \\ 0 \end{bmatrix}$$

The matrices characterizing the polytopes are given as follows

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} E_1 \\ \mathrm{I} \end{bmatrix}, F_2 = \begin{bmatrix} E_2 \\ \mathrm{I} \end{bmatrix}, F_3 = \begin{bmatrix} E_3 \\ \mathrm{I} \end{bmatrix}, F_4 = \begin{bmatrix} E_4 \\ \mathrm{I} \end{bmatrix}$$

$$N_{12} = N_{34} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, N_{14} = N_{23} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$H_{12} = -H_{34} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, H_{14} = -H_{23} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

It is worth noting that the open-loop system is unstable and since solutions with infinite switching in finite time are considered the approach presented in [5] and common Lyapunov based methods are not applicable. The $V-K$ iteration algorithm is initialized using pole placement method. The assigned closed-loop poles for the dynamics in each polytope are $(-3, -2)$ and the corresponding initial controller gains are

$$K_1 = K_3 = \begin{bmatrix} -119.5 \\ -7 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -5 \\ 19.5 \end{bmatrix}^T$$

Using the scheme presented in this paper for a set of constants $\epsilon_{12} = \epsilon_{23} = \epsilon_{14} = \epsilon_{34} = 10$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 100$, the following solutions has been obtained

$$Q_1 = Q_3 = \begin{bmatrix} 135.26 & 2.18 \\ 2.18 & 1.83 \end{bmatrix}, Q_2 = Q_4 = \begin{bmatrix} 84.67 & -5.43 \\ -5.43 & 707.09 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -389.92 \\ -30.14 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -12.88 \\ -0.56 \end{bmatrix}^T$$

$$\gamma_{min} = -4.5328 \times 10^{-4}$$

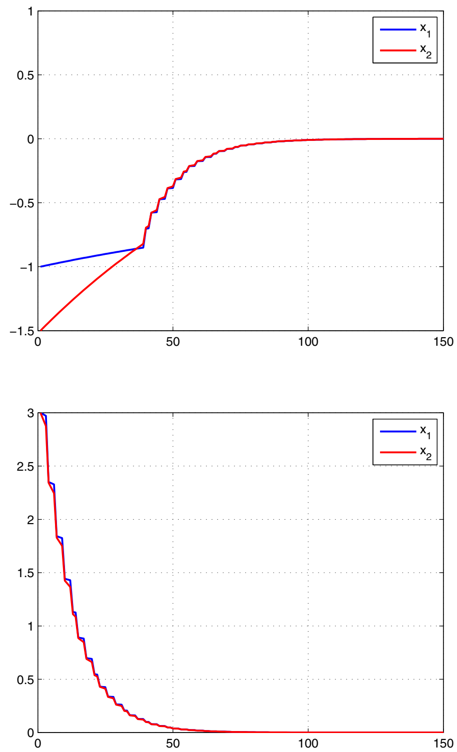

Fig. 2. Evolution of system states when the H_∞ controller is applied: with an initial condition in the interior of a polytope (top) and with an initial condition on a facet (bottom).

for the stable controller synthesis in three iterations and

$$Q_1 = Q_3 = \begin{bmatrix} 463.75 & 24.94 \\ 24.94 & 2.39 \end{bmatrix}, Q_2 = Q_4 = \begin{bmatrix} 52.26 & -7.39 \\ -7.39 & 763.47 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -637.72 \\ -30.14 \end{bmatrix}^T, K_2 = K_4 = \begin{bmatrix} -21.53 \\ -1.69 \end{bmatrix}^T$$

$$\gamma_{min} = -2.7186 \times 10^{-5}$$

for the H_∞ controller design with $\eta = 0.1$ in five iterations. Consequently, it follows from Lemma 4.2 that the closed loop control system is asymptotically stable at the origin and the disturbance attenuation criterion is satisfied. Fig. 1. portrays the simulation results of four different initial conditions (in the absence of disturbance) which prove the stability of the closed loop systems. Notice, in particular, that solutions with infinite switching in finite time on facets are also present (see Fig. 2.). This should be opposed to the results in [5] where only Carathódory solutions are taken into account. Additionally, the simulation results in the presence of disturbance ($v(t) = 4\sin(2\pi t)$) and zero initial conditions are illustrated in Fig. 3. which ascertains the disturbance attenuation performance of the proposed controller.

VI. Conclusions

In this paper, the stability and control problem of PWL and uncertain PWL systems with Filippov Solutions was considered. The foremost purpose of this research was to extend the previous results on PWL systems to the case of solutions with infinite switching in finite time and sliding motions. Correspondingly, a set of matrix inequality conditions was proposed to investigate the stability of a PWL or uncertain PWL system. Additionally, two methods based on BMIs are devised for the synthesis of stable and robust H_∞ controllers for PWL and uncertain PWL systems with Filippov solutions.

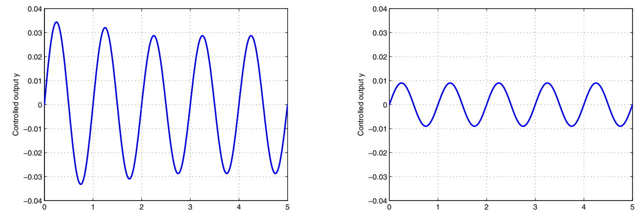

Fig. 3. Response of the closed loop control system with disturbance and zero initial condition: the stable controller synthesis (left) and the H_∞ controller synthesis (right).

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