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Guaranteed Cost H_∞ Controller Synthesis for Switched Systems Defined on Semi-algebraic Sets

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Abstract

A methodology to design guaranteed cost H_∞ controllers for a class of switched systems with polynomial vector fields is proposed. To this end, we use sum of squares programming techniques. In addition, instead of the customary Carathéodory solutions, the analysis is performed in the framework of Filippov solutions which subsumes solutions with infinite switching in finite time and sliding modes. Firstly, conditions assuring asymptotic stability of Filippov solutions pertained to a switched system defined on semi-algebraic sets are formulated. Accordingly, we derive a set of sum of squares feasibility tests leading to a stabilizing switching controller. Finally, we propose a scheme to synthesize stabilizing switching controllers with a guaranteed cost H_∞ disturbance attenuation performance. The applicability of the proposed methods is elucidated thorough simulation analysis.

Keywords: Switched Systems, Filippov Solutions, H_∞ Control, Sum of Squares Programming

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1. Introduction

Over the past decades, there has been a dramatic increase in research on the subject of hybrid systems (see the expository books [1, 2] and the journal articles [3–7]). In particular, a considerable amount of literature has been devoted to the analysis of switched dynamical systems [8–15]. This stems from the fact that switched systems can be more readily described and analyzed by mathematical methodologies [16]. Generally, a switched system can be characterized by a discontinuous differential equation whose right-hand side accepts a family of indexed functions called subsystems. A switched system may possess the state-dependent switching property, if each subsystem is defined on a partition of the state-space; then, a switching occurs whenever the trajectories of the system reach the boundary of these partitions (also known as the switching surface).

Although, form a modeling perspective, switched models are intriguing, the stability and control problem of switched system is not as convenient as their mathematical description (see the survey papers [17, 18]). This stems from the fact that though the dynamics of each subsystem alone is known, the behavior of the trajectories of the overall switched system can be very discrepant. In this regard, Branicky [19] demonstrated that one cannot infer the stability of the overall switched system from the stability of each subsystem. On the other hand, Liberzon [16] evinced that a felicitous switching law may bring about stability, even if all subsystems are unstable. Hespanha [20] proved that if the switches happen *sufficiently* slow, then the stability of the switched system is assured. Furthermore, Leth and Wisniewski [21]

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conceded that a switched system with stable Carathéodory solutions, may possess divergent Filippov solutions, thereby making the overall system unstable. This has shed light on the importance of the considering Filippov solutions when dealing with switched systems. Following the same trend, Ahmadi *et. al.* [22–24] proposed robust stability and control methodologies for switched systems in the context of Filippov solutions.

During the last decade, efficacious techniques for implementing sum of squares (SOS) decomposition of multivariable polynomials using convex optimization has been developed [25, 26]. Owing to its computational efficiency, SOS programming has shown to be virtuous in analyzing a myriad of engineering applications; for instance, [27] suggests an SOS based algorithm for stability and robustness analysis of nonlinear systems via contraction metrics; [28] considered a nonlinear control method using robust SOS scheme for uncertain hypersonic aircrafts; [29] is concerned with a model predictive control strategy for input saturated polynomial systems using SOS programming; [30] devises an SOS optimization based design algorithm for halfband product filters for orthogonal wavelets; [31] brings forward a set of SOS feasibility tests, which determine the stability of an uncertain gene regulatory network; [32] proposes an SOS programming based method to calculate the robust observability and controllability degree of linear time invariant systems with semi-algebraic uncertainty.

Previously, we have addressed the stability issue of nonlinear switched systems defined on compact sets in the context of Filippov solutions [22], based on mathematical tools from the theory of differential inclusions. Additionally, we demonstrated that if the subsystems consist of polynomial vector

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fields and if the subsystems are defined on semi-algebraic sets, the stability analysis can be efficiently carried out using SOS programming. In the present paper, we propose a guaranteed cost H_∞ controller synthesis methodology for switched systems defined on semi-algebraic sets established upon SOS programming. We first bring forward SOS feasibility tests which determine a stabilizing switching controller. Subsequently, we formulate conditions based on SOS programming, which provide a switching controller with guaranteed cost H_∞ performance. These latter conditions include bilinear terms, for which an appropriate iterative algorithm is devised. The validity of the proposed methods is verified through a numerical example, as well.

The rest of the paper is organized as follows. The subsequent section includes the notations and a number of preliminary definitions. Section 3 examines the problem of stabilizing switching controllers for switched systems defined on semi-algebraic sets. In Section 4, the gauranteed cost H_∞ controller synthesis scheme is outlined. Section 5 discusses the applicability of the proposed method through a simulation example. Finally, Section 6 concludes the paper.

2. Notations and Definitions

The notations adopted in this paper are described next. The set of positive real numbers is denoted by $\mathbb{R}_{\geq 0}$, the Euclidean vector norm on $\mathbb{R}^n$ by $\| \cdot \|$, the inner product by $\langle \cdot \rangle$, and the closed ball of radius ϵ in $\mathbb{R}^n$ centered at origin by $\mathcal{B}_\epsilon^n$. Denote by $\mathcal{P}_{p \times q}(x)$ the class of $p \times q$ polynomial matrices in x, and we drop the subscript when we simply mean the class of polynomial functions. Accordingly, $\mathcal{P}_{sos}(x) \subset \mathcal{P}(x)$ signifies the class of polynomial

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functions in x with an SOS decomposition; i.e., $p(x) \in \mathcal{P}_{sos}(x)$ if and only if there are $p_i(x) \in \mathcal{P}(x), i \in \{1, \dots, k\}$ such that $p(x) = p_i^2(x) + \dots + p_k^2(x)$. In addition, by a symmetric polynomial map $Q(x) \in \mathcal{P}_{n \times n}(x)$, we imply that $Q(x)$ is symmetric for all x. We denote the interior of a set K by $int(K)$, and the boundary of K by $bd(K)$; then, $K = int(K) \cup bd(K)$. Furthermore, the closed convex hull of the set K is signified by $co(K)$, and the set of all subsets of K (power set of K) by 2^K. For a continuously differentiable ($\mathcal{C}^1$) function $V(x)$, $\frac{\partial V(x)}{\partial x}$ designates the column vector with first-order partial derivatives of $V(x)$.

Consider the class of n-dimensional nonlinear switched systems $\mathcal{S} = \{\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{W}, X, I, \mathscr{F}, \mathscr{G}\}$ wherein $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{Y} \subset \mathbb{R}^m$, $\mathcal{U} \subset \mathbb{R}^{n_u}$, and $\mathcal{W} \subset \mathbb{R}^{n_w}$ are compact subsets of Euclidean spaces and define respectively the state space, the output space, the input space and the disturbance space. $X = \{X_i\}_{i \in I}$ is the set containing (closed) partitions of $\mathcal{X}$ with index set $I = \{1, 2, 3, \dots, N\}$. $\mathscr{F} = \{F_i\}_{i \in I}$ and $\mathscr{G} = \{G_i\}_{i \in I}$ are families of smooth functions. Each of the functions F_i and G_i are determined respectively by a triplet $\{A_i(x), B_i(x), G_i(x)\}$ and a bi-tuple $\{C_i(x), D_i(x)\}$. Furthermore, $F_i : P_i \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^n; (x, u, w) \mapsto \{z \in \mathbb{R}^n \mid z = A_i(x)x + B_i(x)u + G_i(x)w\}$ and $G_i : P_i \times \mathcal{U} \to \mathbb{R}^m; (x, u) \mapsto \{z \in \mathbb{R}^m \mid z = C_i(x)x + D_i(x)u\}$ with P_i an open neighborhood of X_i. Note that $\mathcal{X} = \bigcup_{i \in I} X_i$, and $int(X_i) \neq \emptyset$ for all $X_i \in X$. Also, denote $\tilde{I} = \{(i, j) \in I \times I \mid X_i \cap X_j \neq \emptyset, i \neq j\}$ the set of index pairs which determines the partitions with non-empty intersections; hence, $X_i \cap X_j = bd(X_i) \cup bd(X_j)$ for all $(i, j) \in \tilde{I}$. We further assume that each function $w(t)$ belongs to $\mathcal{L}_2[0, \infty)$; i.e., the class of functions for which

$$\|w\|_{\mathcal{L}_2} = \left(\int_0^\infty w(t)^T w(t) \, \mathrm{d}t \right)^{\frac{1}{2}}$$

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is well-defined and finite.

Define the set valued maps $\mathcal{F}$ and $\mathcal{G}$ as

$$\mathcal{F} : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to 2^{\mathbb{R}^n}; (x, u, w) \mapsto \{v \in \mathbb{R}^n \mid v = F_i(x, u, w) \text{ if } x \in X_i\} \quad (1)$$

$$\mathcal{G} : \mathcal{X} \times \mathcal{U} \to 2^{\mathbb{R}^m}; (x, u) \mapsto \{v \in \mathbb{R}^m \mid v = G_i(x, u) \text{ if } x \in X_i\} \quad (2)$$

Then, the global dynamics of the switched system is described by the following differential inclusions

$$\dot{x}(t) \quad \in \quad \mathcal{F}\big(x(t), u(t), w(t)\big) \tag{3}$$

$$\dot{x}(t) \quad \in \quad co\bigg(\mathcal{F}\big(x(t), u(t), w(t)\big)\bigg) \tag{4}$$

$$y(t) \quad \in \quad \mathcal{G}(x(t), u(t)) \tag{5}$$

The choice of whether differential inclusion (3) or (4) describe the dynamics of the switched system depends on the nature of the motion to be considered. By a Carathéodory solution of differential inclusion (3) at $\zeta_0 \in \mathcal{X}$, we understand an absolutely continuous function $[0, T) \to \mathcal{X}; t \mapsto \zeta(t)$ $(T > 0)$ which solves the following Cauchy problem

$$\dot{\zeta}(t) \in \mathcal{F}\big(\zeta(t), u(t), w(t)\big), \quad \text{a.e.,} \quad \zeta(0) = \zeta_0$$

A Filippov solution to differential inclusion (3) at $\zeta_0 \in \mathcal{X}$ is an absolutely continuous function $[0, T) \to \mathcal{X}; t \mapsto \zeta(t)$ which solves the following Cauchy problem

$$\dot{\zeta}(t) \in co\bigg(\mathcal{F}\big(\zeta(t), u(t), w(t)\big)\bigg), \quad \text{a.e.,} \quad \zeta(0) = \zeta_0$$

That is, an absolutely continuous solution to (4) with $\zeta_0 \in \mathcal{X}$.

Our objective is to design a switching controller

$$u(x) \in \mathcal{K}(x)$$

$$\mathcal{K} : \mathbb{R}^n \to 2^{\mathcal{U}}; x \mapsto \{s \in \mathcal{U} \mid s = K_i(x)x \quad \text{if} \quad x \in X_i\} \tag{6}$$

such that in addition to asymptotic stability of the Filippov solutions of (3) assures that, for some $\eta > 0$ and a function $W(x)$ (with some properties that will be discussed later in this paper, e.g., $W(0) = 0$), the following performance is always satisfied

$$\|y\|_{\mathcal{L}_2}^2 \leq \eta^2 \|w\|_{\mathcal{L}_2}^2 + W(x_0) \tag{7}$$

Then, under zero initial conditions, the system is said to possess H_∞ disturbance attenuation performance η.

In case of matrix inequalities, I denotes the unity matrix (the size of I can be inferred from the context) and should be distinguished from the index set I. In matrices, $\star$ in place of a matrix entry a_{mn} means that $a_{mn} = a_{nm}{}^T$. By $A^l(x)$, we denote the l'th row of a polynomial matrix $A(x)$.

Let $L_i = \{l_1, l_2, \ldots, l_{s_i}\}$ be the set of row indices of $H_i(x) \equiv \begin{bmatrix} B_i(x) & G_i(x) \end{bmatrix}$ whose corresponding rows are zero, i.e., $H_i^{l_j}(x) = 0$, for $j = 1, \ldots l_{s_i}$. Furthermore, for L_i defined as above, we use the notation $\hat{x}^i = (x_{l_1}, x_{l_2}, \ldots, x_{l_{s_i}})$.

We will be explicitly concerned with switched systems defined on semi-algebraic sets. In this case, each partition is determined by a semi-algebraic set

$$X_i = \big\{ x \in \mathcal{X} \mid \chi_i(x) = 0, \ \xi_{ik}(x) \geq 0 \quad \text{for} \quad k \in N_i \big\}, \quad i \in I \tag{8}$$

and each boundary is characterized by a variety

$$X_i \cap X_j = \big\{ x \in \mathcal{X} \mid \chi_{ij}(x) = 0 \big\}, \quad (i, j) \in \tilde{I} \tag{9}$$

Remark that $\chi_i(x)$ could take the structure $\chi_i(x) = p_1^2(x) + p_2^2(x) + \cdots + p_n^2(x)$; as a consequence, $\chi_i(x) = 0$ implies $p_1(x) = p_2(x) = \cdots = p_n(x) = 0$.

In the sequel, we employ the next lemma from [33].

Lemma 1 ([33]). *Denote by $\bigotimes$ the Kronecker product. Suppose $F(x) \in \mathcal{P}_{n \times n}(x)$ is symmetric and of degree $2d$ for all $x \in \mathbb{R}^n$. In addition, let $Z(x) \in \mathcal{P}_{n \times 1}(x)$ be a column vector of monomials of degree no greater than d and consider the following conditions*

(A) $F(x) \geq 0$ for all $x \in \mathbb{R}^n$

(B) $v^T F(x)v \in \mathcal{P}_{sos}(x, v)$, for any $v \in \mathbb{R}^n$.

(C) There exists a positive semi-definite matrix Q such that $v^T F(x)v = (v \bigotimes Z(x))^T Q(v \bigotimes Z(x))$, for any $v \in \mathbb{R}^n$.

Then $(A) \Longleftarrow (B)$ and $(B) \Longleftrightarrow (C)$.

3. Stabilizing State-Feedback Controller Synthesis

In this section, we first derive a stability theorem for the class of switched systems expounded in Section 2. Then, we formulate SOS conditions which ensure the existence of a stabilizing state feedback law in the form of (6). At this point, let us construct the following switched Lyapunov function

$$\Phi : \mathcal{X} \to 2^{\mathbb{R}}; x \mapsto \{z \in \mathbb{R} \mid z = V_i(x), \text{ if } x \in X_i\}, \tag{10}$$

where

$$V_i(x) = x^T Q_i(\hat{x}^i)x$$

and such that

$$V_i(x) = V_j(x) \quad \text{for all} \quad x \in bd(X_i) \cup bd(X_j) \quad \text{and all} \quad (i,j) \in \tilde{I}. \tag{11}$$

Then, Φ is single-valued and locally Lipschitzean at any point $x \in \mathcal{X}$ (see Fig. 1 for an illustration).
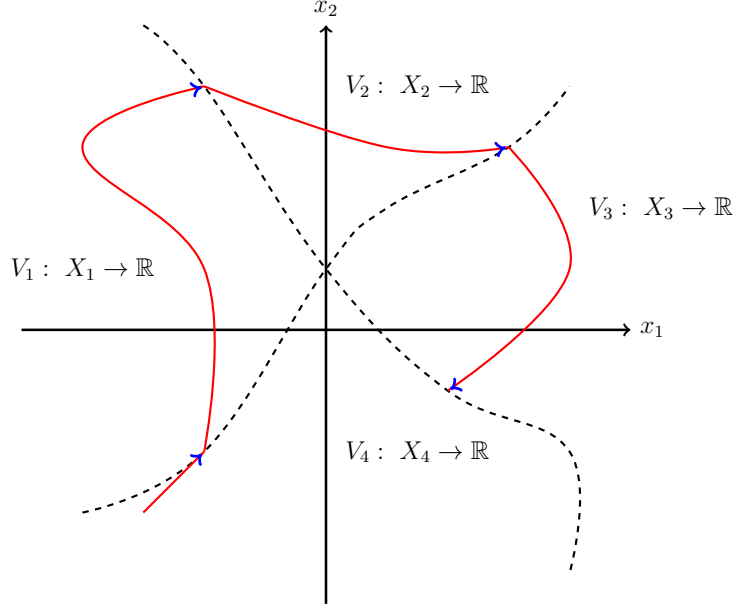
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Figure 1: At each partition of the state space X_1–X_4, a $\mathcal{C}^1$ function V_1–V_4 is assigned.

Theorem 1. *Consider the a switched system $\mathcal{S}$ with Filippov solutions described by (3)–(4). Suppose $u \equiv 0$ and $w \equiv 0$, and let v be an arbitrary vector of dimension n. If there exist a family of symmetric polynomial maps $\{Q_i(\hat{x}^i)\}_{i \in I} \subset \mathcal{P}_{n \times n}(\hat{x}^i)$, constants $\epsilon_i > 0$, and $\{q_{ik}(x,v)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x,v)$, $\{w_{ik}(x,v)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x,v)$ whose degree in v are equal to two , and $\varepsilon_i(x,v) \in \mathcal{P}(x,v)$, $\rho_i(x,v) \in \mathcal{P}(x,v)$, $p_{ij}(x,v) \in \mathcal{P}(x,v)$, and $r_{ij}(x,v) \in \mathcal{P}(x,v)$ whose degree in v are two with $i \in I$ and $(i,j) \in \tilde{I}$, such that*

$$v^T \left(Q_i(\hat{x}^i) - \epsilon_i \mathrm{I} \right) v - \varepsilon_i(x,v)\chi_i(x) - \sum_{k \in N_i} q_{ik}(x,v)\xi_{ik}(x) \in \mathcal{P}_{sos}(x,v) \quad (12)$$

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$$-v^T \left(A_i(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) A_i(x) + \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_i^l(x) x \right) \right) v$$
$$-\rho_i(x,v)\chi_i(x) - \sum_{k \in N_i} w_{ik}(x,v)\xi_{ik}(x) \in \mathcal{P}_{sos}(x,v) \qquad (13)$$

for all $i \in I$ and

$$-v^T \left(A_j(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) A_j(x) + \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_j^l(x) x \right) \right) v$$
$$-r_{ij}(x,v)\chi_{ij}(x) \in \mathcal{P}_{sos}(x,v) \qquad (14)$$

$$v^T Q_i(\hat{x}^i) v + p_{ij}(x,v)\chi_{ij}(x) = v^T Q_j(\hat{x}^j) v \qquad (15)$$

for all $(i,j) \in \tilde{I}$. Then, all of the Filippov solutions of (4) with $u \equiv 0$ and $w \equiv 0$ are asymptotically stable at the origin.

Proof. This is a direct result of applying Proposition 10 and Proposition 12 in [22]. Note that the additional terms $-\sum_{k \in N_i} q_{ik}(x,v)\xi_{ik}(x)$ and $-\sum_{k \in N_i} w_{ik}(x,v)\xi_{ik}(x)$ are derived from applying the generalized S-procedure (see Section 3.3 in [22]), and $-r_{ij}(x,v)\gamma_{ij}(x)$, $-\varepsilon_i(x,v)\chi_i(x)$, and $-\rho_i(x,v)\chi_i(x)$ are obtained using the Finsler's Lemma. Condition (15) ensures that the local Lyapunov functions are continuous along the boundaries. Conditions (12)-(14) also correspond to (34)–(36) in [22]. $\qquad \square$

Remark 1. The assumption on Q_i to be only a function of $\hat{x}_i$ is to avoid nonlinear terms involved when computing the derivative of Q_i with respect to x.

Applying the proposed switching controller (6) to the system given by (3)-(5) yields the controlled system $\tilde{S}$ given by

$$\dot{x}(t) \in \tilde{\mathcal{F}}\big(x(t), w(t)\big) \qquad (16)$$

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$$\dot{x}(t) \in co\left(\tilde{\mathcal{F}}\big(x(t), w(t)\big)\right) \tag{17}$$

$$y(t) \in \tilde{\mathcal{G}}(x(t)) \tag{18}$$

where

$$\tilde{\mathcal{F}} : \mathcal{X} \times \mathcal{W} \to 2^{\mathbb{R}^n}; (x, w) \mapsto \{v \in \mathbb{R}^n \mid v = A_{ci}(x)x + G_i(x)w \text{ if } x \in X_i\} \tag{19}$$

$$\tilde{\mathcal{G}} : \mathcal{X} \to 2^{\mathbb{R}^m}; x \mapsto \{v \in \mathbb{R}^m \mid v = C_{ci}(x)x \text{ if } x \in X_i\} \tag{20}$$

with

$$
\begin{aligned}
A_{ci}(x) &= A_i(x) + B_i(x)K_i(x) \\
C_{ci}(x) &= C_i(x) + D_i(x)K_i(x)
\end{aligned}
\tag{21}
$$

If we supplant A_i with A_{ci} in the Theorem 1, we arrive at the following conclusion.

Corollary 1. *Consider the controlled switched system with Filippov solutions $\tilde{\mathcal{S}}$ as defined by (16) and (17). Assume $w \equiv 0$. Let v be an arbitrary vectors of dimension n. Define*

$$
\begin{aligned}
\Psi_i(x) = A_i(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i)A_i(x) + \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l}\big(A_i^l(x)x\big) \\
+ K_i(x)^T B_i(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i)B_i(x)K_i(x)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\Psi_{ij}(x) = A_j(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i)A_j(x) + \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l}\big(A_j^l(x)x\big) \\
+ K_j(x)^T B_j(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i)B_j(x)K_j(x)
\end{aligned}
\tag{23}
$$

If there exist a family of symmetric $\{Q_i(\hat{x}^i)\}_{i \in I} \subset \mathcal{P}_{n \times n}(\hat{x}^i)$, polynomial matrices $\{K_i(x)\}_{i \in I} \subset \mathcal{P}_{n_u \times n}(x)$, constants $\epsilon_i > 0$, and $\{q_{ik}(x, v)\}_{k \in N_i} \subset$

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$\mathcal{P}_{sos}(x,v)$, $\{w_{ik}(x,v)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x,v)$ *whose degree in* v *are equal to two ,* *and* $\varepsilon_i(x,v) \in \mathcal{P}(x,v)$, $\rho_i(x,v) \in \mathcal{P}(x,v)$, $p_{ij}(x,v) \in \mathcal{P}(x,v)$, *and* $r_{ij}(x,v) \in \mathcal{P}(x,v)$ *whose degree in* v *are two with* $i \in I$ *and* $(i,j) \in \tilde{I}$, *such that*

$$v^T \left(Q_i(\hat{x}^i) - \epsilon_i I \right) v - \varepsilon_i(x,v)\chi_i(x) - \sum_{k \in N_i} q_{ik}(x,v)\xi_{ik}(x) \in \mathcal{P}_{sos}(x,v) \quad (24)$$

$$-v^T \Psi_i(x)v - \rho_i(x,v)\chi_i(x) - \sum_{k \in N_i} w_{ik}(x,v)\xi_{ik}(x) \in \mathcal{P}_{sos}(x,v) \quad (25)$$

for all $i \in I$ *and*

$$-v^T \Psi_{ij}(x)v - r_{ij}(x,v)\chi_{ij}(x) \in \mathcal{P}_{sos}(x,v) \quad (26)$$

$$v^T Q_i(\hat{x}^i)v + p_{ij}(x,v)\chi_{ij}(x) = v^T Q_j(\hat{x}^j)v \quad (27)$$

for all $(i,j) \in \tilde{I}$. *Then, the switching controller synthesis* (6) *renders the origin asymptotically stable.*

However, it can be readily discerned that (25) and (26) include bilinear terms including multiplication of the variables $K_i(x)$ and $Q_i(\hat{x}^i)$. We will not treat this problem here and defer it until the end of the next section.

4. Guaranteed Cost H_∞ Controller Synthesis

At this juncture, we are prepared to assert the main results of this paper; i.e., to characterize a set of SOS conditions to ensure that, in addition to asymptotic stability of the Filippov solutions, the performance objective (7) is satisfied. Assume $i_0 \in I$ is the index satisfying $x_0 \in X_{i_0}$.

Theorem 2. *Let $\tilde{\mathcal{S}}$ be the controlled switched system with Filippov solutions described by (16)–(18). Let v_1 and v_2 be two arbitrary vectors of dimension n and $2n$, respectively. Suppose*

$$
\begin{aligned}
\Xi_i(x) = A_{ci}(x)^T Q_i(\hat{x}^i) \;\; & + \;\; Q_i(\hat{x}^i) A_{ci}(x) \\
& + \;\; C_{ci}(x)^T C_{ci}(x) + \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_{ci}^l(x) x \right) \quad (28)
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{ij}(x) = A_{cj}(x)^T Q_i(\hat{x}^i) \;\; & + \;\; Q_i(\hat{x}^i) A_{cj}(x) \\
& + \;\; C_{cj}(x)^T C_{cj}(x) + \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_{cj}^l(x) x \right) \quad (29)
\end{aligned}
$$

For a given $\eta > 0$, if there exist a family of symmetric polynomial maps $\{Q_i(\hat{x}^i)\}_{i \in I} \subset \mathcal{P}_{n \times n}(\hat{x}^i)$, constants $\epsilon_i > 0$, and $\{q_{ik}(x, v_1)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x, v_1)$, $\{w_{ik}(x, v_2)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x, v_2)$ whose degree in v_1 and v_2 are equal to two, and $\varepsilon_i(x, v_1) \in \mathcal{P}(x, v_1)$, $\rho_i(x, v_2) \in \mathcal{P}(x, v_2)$, $p_{ij}(x, v_1) \in \mathcal{P}(x, v_1)$, and $r_{ij}(x, v_2) \in \mathcal{P}(x, v_2)$ whose degree in v_1 and v_2 are two with $i \in I$ and $(i, j) \in \tilde{I}$, such that

$$
v_1^T \left(Q_i(\hat{x}^i) - \epsilon_i I \right) v_1 - \varepsilon_i(x, v_1) \chi_i(x) - \sum_{k \in N_i} q_{ik}(x, v_1) \xi_{ik}(x) \in \mathcal{P}_{sos}(x, v_1) \quad (30)
$$

$$
\begin{aligned}
-v_2^T \begin{bmatrix} \Xi_i(x) & Q_i(\hat{x}^i) G_i(x)^T G_i(x) \\ \star & -\eta^2 G_i(x)^T G_i(x) \end{bmatrix} v_2 \;\; & - \;\; \rho_i(x, v_2) \chi_i(x) \\
& - \;\; \sum_{k \in N_i} w_{ik}(x, v_2) \xi_{ik}(x) \in \mathcal{P}_{sos}(x, v_2)
\end{aligned}
$$

$$
(31)
$$

for all $i \in I$ and

$$
-v_2^T \begin{bmatrix} \Xi_{ij}(x) & Q_i(\hat{x}^i) G_j(x)^T G_j(x) \\ \star & -\eta^2 G_j(x)^T G_j(x) \end{bmatrix} v_2 - r_{ij}(x, v_2) \gamma_{ij}(x) \in \mathcal{P}_{sos}(x, v_2)
$$

$$
(32)
$$

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$$v_1^T Q_i(\hat{x}^i) v_1 + p_{ij}(x, v_1) \chi_{ij}(x) = v_1^T Q_j(\hat{x}^j) v_1 \tag{33}$$

for all $(i,j) \in \tilde{I}$. Then, the Filippov solutions of the switched system $\tilde{S}$ are asymptotically stable at origin. Furthermore, the performance criterion (7) is satisfied with $W(x_0) = x_0^T Q_{i_0}(\hat{x}_0^i) x_0$.

Proof. (33) implies (11), and (30) is equivalent to (12). Utilizing Shur's complement theorem and removing the auxiliary terms from the Generalized S-procedure, (31) and (32) reduces to

$$-v^T \Bigg(A_{ci}(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) A_{ci}(x) + \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \big(A_{ci}^l(x) x \big)$$

$$+ C_{ci}(x)^T C_{ci}(x) + \eta^{-2} Q_i(\hat{x}^i) G_i(x)^T G_i(x) Q_i(\hat{x}^i) \Bigg) v < 0 \tag{34}$$

for all $x \in int(X_i)$ with $i \in I$ and

$$-v^T \Bigg(A_{cj}(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) A_{cj}(x) + \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \big(A_{cj}^l(x) x \big)$$

$$+ C_{cj}(x)^T C_{cj}(x) + \eta^{-2} Q_i(\hat{x}^i) G_j(x)^T G_j(x) Q_i(\hat{x}^i) \Bigg) v < 0 \tag{35}$$

for all $x \in bd(X_i) \cup bd(X_j)$ with $(i,j) \in \tilde{I}$. One can deduce that if the above conditions hold so does (13) and (14). Hence, the Filippov solutions of (16) are asymptotically stable at origin. Differentiating and integrating Φ with

respect to t yields

$$
\begin{aligned}
\int_0^\infty \frac{\mathrm{d}\Phi}{\mathrm{d}t}\mathrm{d}t \;=\;\;
& \int_0^{t_1}\bigg[\left(w^T G_1(x)^T + x^T A_{c1}(x)^T\right) Q_1(\hat{x}^1)x \\
& +x^T \sum_{l\in L_1}\frac{\partial Q_1(\hat{x}^1)}{\partial x_l}\left(A_{c1}^l(x)x\right)x \\
& +x^T Q_1(\hat{x}^1)\left(A_{c1}(x)x + G_1(x)w\right)\bigg]\mathrm{d}t + \ldots \\
& +\int_{t_1}^{t_2}\bigg[\left(w^T G_2(x)^T + x^T A_{c2}(x)^T\right) Q_2(\hat{x}^2)x \\
& +x^T \sum_{l\in L_2}\frac{\partial Q_2(\hat{x}^2)}{\partial x_l}\left(A_{c2}^l(x)x\right)x \\
& +x^T Q_2(\hat{x}^2)\left(A_{c2}(x)x + G_2(x)w\right)\bigg]\mathrm{d}t + \ldots \\
& +\sum_{j=1}^{r}\alpha_j\bigg\{\int_{t_{k-1}}^{t_k}\bigg[\left(w^T G_j(x)^T + x^T A_{cj}(x)^T\right) Q_k(\hat{x}^k)x \\
& +x^T \sum_{l\in L_j}\frac{\partial Q_k(\hat{x}^k)}{\partial x_l}\left(A_{cj}^l(x)x\right)x \\
& +x^T Q_k(\hat{x}^k)\left(A_{cj}(x)x + G_j(x)w\right)\bigg]\mathrm{d}t\bigg\} + \ldots \\
& +\sum_{j=1}^{m}\beta_j\bigg\{\int_{t_{h-1}}^{t_h}\bigg[\left(w^T G_j(x)^T + x^T A_{cj}(x)^T\right) Q_h(\hat{x}^h)x \\
& +x^T \sum_{l\in L_j}\frac{\partial Q_h(\hat{x}^h)}{\partial x_l}\left(A_{cj}^l(x)x\right)x \\
& +x^T Q_h(\hat{x}^h)\left(A_{cj}(x)x + G_j(x)w\right)\bigg]\mathrm{d}t\bigg\} + \ldots \\
& +\int_{t_n}^{\infty}\bigg[\left(w^T G_n(x)^T + x^T A_{cn}(x)^T\right) Q_n(\hat{x}^n)x \\
& +x^T \sum_{l\in L_n}\frac{\partial Q_n(\hat{x}^n)}{\partial x_l}\left(A_{cn}^l(x)x\right)x \\
& +x^T Q_n(\hat{x}^n)\left(A_{cn}(x)x + G_n(x)w\right)\bigg]\mathrm{d}t
\end{aligned}
\tag{36}
$$

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wherein $\alpha_j, \beta_j > 0$ such that $\sum_{j=1}^n \alpha_j = 1$, and $\sum_{j=1}^n \beta_j = 1$. m and r are the number of neighboring cells to a boundary where the solutions possess infinite switching in finite time (in the time intervals of $[t_{k-1}, t_k]$ and $[t_{h-1}, t_h]$), respectively. In the above formulations, without loss of generality, it is assumed that the initial conditions are located in the partition X_1 and the motion includes both the interior of partitions as well as their boundaries. For any $x \in X_i$ from (34), it follows that

$$
\int_{t_{i-1}}^{t_i} \Bigg[\left(w^T G_i(x)^T + x^T A_{ci}(x)^T \right) Q_i(\hat{x}^i) x
$$

$$
+ x^T \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_{ci}^l(x) x \right) x
$$

$$
+ x^T Q_i(\hat{x}^i) \left(A_{ci}(x) x + G_i(x) w \right) \Bigg] \mathrm{d}t
$$

$$
= \int_{t_{i-1}}^{t_i} \Bigg[x^T \left(A_{ci}(x)^T Q_i(\hat{x}^1) + Q_i(\hat{x}^i) A_{ci}(x) + \sum_{l \in L_i} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \left(A_{ci}^l(x) x \right) x \right) x
$$

$$
+ x^T Q_i(\hat{x}^i) G_i(x) w + w^T G_i(x)^T Q_i(\hat{x}^i) x \Bigg] \mathrm{d}t
$$

$$
< \int_{t_{i-1}}^{t_i} \Bigg[x^T \left(-C_{ci}(x)^T C_{ci}(x) - \eta^{-2} Q_i(\hat{x}^i) G_i(x)^T G_i(x) Q_i(\hat{x}^i) \right) x
$$

$$
+ x^T Q_i(\hat{x}^i) G_i(x) w + w^T G_i(x)^T Q_i(\hat{x}^i) x \Bigg] \mathrm{d}t
$$

$$
= \int_{t_{i-1}}^{t_i} \Bigg[- y^T y + x^T Q_i(\hat{x}^i) G_i(x) w + w^T G_i(x)^T Q_i(\hat{x}^i) x
$$

$$
- \eta^{-2} x^T \left(Q_i(\hat{x}^i) G_i(x)^T G_i(x) Q_i(\hat{x}^i) \right) x
$$

$$
+ \left(\eta w - \eta^{-1} G_i(x) Q_i(\hat{x}^i) x \right)^T \left(\eta w - \eta^{-1} G_i(x) Q_i(\hat{x}^i) x \right)
$$

$$
- \left(\eta w - \eta^{-1} G_i(x) Q_i(\hat{x}^i) x \right)^T \left(\eta w - \eta^{-1} G_i(x) Q_i(\hat{x}^i) x \right) \Bigg] \mathrm{d}t
$$

$$
\leq \int_{t_{i-1}}^{t_i} \left[-y^T y + \eta^2 w^T w \right] \mathrm{d}t
$$

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Correspondingly,

$$\sum_{k=1}^{n} \alpha_k \Bigg\{ \int_{t_{j-1}}^{t_j} \Bigg[\left(w^T G_k(x)^T + x^T A_{ck}(x)^T \right) Q_j(\hat{x}^j) x$$

$$+ x^T \sum_{l \in L_k} \frac{\partial Q_j(\hat{x}^j)}{\partial x_l} \left(A_{ck}^l(x) x \right) x$$

$$+ x^T Q_j(\hat{x}^j) \left(A_{ck}(x) x + G_k(x) w \right) \Bigg] \mathrm{d}t$$

$$= \sum_{k=1}^{n} \alpha_k \Bigg\{ \int_{t_{j-1}}^{t_j} \Bigg[x^T \left(A_{ck}(x)^T Q_j(\hat{x}^1) + Q_j(\hat{x}^i) A_{ck}(x) + \sum_{l \in L_k} \frac{\partial Q_j(\hat{x}^i)}{\partial x_l} \left(A_{ck}^l(x) x \right) x \right) x$$

$$+ x^T Q_j(\hat{x}^j) G_k(x) w + w^T G_k(x)^T Q_j(\hat{x}^i) x \Bigg] \mathrm{d}t \Bigg\}$$

$$< \sum_{k=1}^{n} \alpha_k \Bigg\{ \int_{t_{j-1}}^{t_j} \Big[x^T \left(-C_k(x)^T C_k(x) - \eta^{-2} Q_j(\hat{x}^j) G_k(x)^T G_k(x) Q_j(\hat{x}_j) \right) x$$

$$+ x^T Q_j(\hat{x}^i) G_k(x) w + w^T G_k(x)^T Q_j(\hat{x}^i) x \Big] \mathrm{d}t \Bigg\}$$

$$\leq \int_{t_{j-1}}^{t_j} \Bigg[-y^T y - \sum_{k=1}^{n} \alpha_k \Big\{ \eta^{-2} x^T Q_j(\hat{x}^j) G_k(x)^T G_k(x) Q_j(\hat{x}_j) x$$

$$+ \left(\eta w - \eta^{-1} G_k(x) Q_j(\hat{x}^j) x \right)^T \left(\eta w - \eta^{-1} G_k(x) Q_j(\hat{x}^j) x \right)$$

$$- \left(\eta w - \eta^{-1} G_k(x) Q_j(\hat{x}^j) x \right)^T \left(\eta w - \eta^{-1} G_k(x) Q_j(\hat{x}^j) x \right) \Big\} \Bigg] \mathrm{d}t$$

$$\leq \int_{t_{j-1}}^{t_j} \left[-y^T y + \eta^2 w^T w \right] \mathrm{d}t$$

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From the above calculations, it can be discerned that

$$\int_0^\infty \frac{\mathrm{d}\Phi}{\mathrm{d}t} \mathrm{d}t \ \leq \ \int_0^{t_1} (-y^T y + \eta^2 w^T w) \mathrm{d}t$$
$$+ \int_{t_1}^{t_2} (-y^T y + \eta^2 w^T w) \mathrm{d}t + \dots$$
$$+ \int_{t_{k-1}}^{t_k} (-y^T y + \eta^2 w^T w) \mathrm{d}t + \dots$$
$$+ \int_{t_{h-1}}^{t_h} (-y^T y + \eta^2 w^T w) \mathrm{d}t + \dots$$
$$+ \int_{t_n}^\infty (-y^T y + \eta^2 w^T w) \mathrm{d}t$$

which is simplified to

$$\Phi\big(x(\infty)\big) - \Phi\big(x(0)\big) \leq \int_0^\infty (-y^T y + \eta^2 w^T w) \mathrm{d}t$$

Additionally, because the Filippov solutions of the system are asymptotically stable at origin (as demonstrated earlier in this proof), we have $x(\infty) = 0$. Thus,

$$\Phi(x_0) \leq -\|y\|_{\mathcal{L}_2}^2 + \|w\|_{\mathcal{L}_2}^2$$

which with little manipulation is equivalent to (7) with $W(x_0) = \Phi(x_0) = x_0 Q_{i_0}(\hat{x}_0^{i_0}) x_0$. Consequently, the performance objective is also satisfied. $\square$

The subsequent corollary brings forward SOS programming conditions to synthesize a switching controller satisfying our stability and performance objectives.

Corollary 2. *Let $\tilde{S}$ be a switched system with Filippov solutions subject to disturbance $w(t) \in \mathcal{L}_2[0, \infty)$ as given by (16)–(18). Let v_1 and v_2 be two*

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arbitrary vectors of dimension n and 3n, respectively. Define

$$
\begin{aligned}
\Gamma_i(x) \ = \ & \Xi_i(x) + K_i(x)^T B_i(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) B_i(x) K_i(x) \\
& + C_i(x)^T D_i(x) K_i(x) + K_i(x)^T D_i(x)^T C_i(x) \quad\quad (37)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{ij}(x) \ = \ & \Xi_{ij}(x) + K_j(x)^T B_j(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) B_j(x) K_j(x) \\
& + C_j(x)^T D_j(x) K_j(x) + K_j(x)^T D_j(x)^T C_j(x) \quad\quad (38)
\end{aligned}
$$

Given a constant $\eta > 0$, if there exist a family of symmetric polynomial maps $\{Q_i(\hat{x}^i)\}_{i \in I} \subset \mathcal{P}_{n \times n}(\hat{x}^i)$, constants $\epsilon_i > 0$, and $\{q_{ik}(x, v_1)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x, v_1)$, $\{w_{ik}(x, v_2)\}_{k \in N_i} \subset \mathcal{P}_{sos}(x, v_2)$ whose degree in v_1 and v_2 are equal to two, and $\varepsilon_i(x, v_1) \in \mathcal{P}(x, v_1)$, $\rho_i(x, v_2) \in \mathcal{P}(x, v_2)$, $p_{ij}(x, v_1) \in \mathcal{P}(x, v_1)$, and $r_{ij}(x, v_2) \in \mathcal{P}(x, v_2)$ whose degree in v_1 and v_2 are two with $i \in I$ and $(i, j) \in \tilde{I}$, such that

$$
v_1^T \left(Q_i(\hat{x}^i) - \epsilon_i I \right) v_1 - \varepsilon_i(x, v_1) \chi_i(x) - \sum_{k \in N_i} q_{ik}(x, v_1) \xi_{ik}(x) \in \mathcal{P}_{sos}(x, v_1) \quad (39)
$$

$$
\begin{aligned}
- v_2^T & \begin{bmatrix} \Gamma_i(x) & Q_i(\hat{x}^i) G_i(x)^T G_i(x) & K_i(x)^T D_i(x)^T \\ \star & -\eta^2 G_i(x)^T G_i(x) & 0 \\ \star & \star & -I \end{bmatrix} v_2 \\
& - \rho_i(x, v_2) \chi_i(x) - \sum_{k \in N_i} w_{ik}(x, v_2) \xi_{ik}(x) \in \mathcal{P}_{sos}(x, v_2) \quad\quad (40)
\end{aligned}
$$

for all $i \in I$ and

$$
\begin{aligned}
- v_2^T & \begin{bmatrix} \Gamma_{ij}(x) & Q_i(\hat{x}^i) G_j(x)^T G_j(x) & K_j(x)^T D_j(x)^T \\ \star & -\eta^2 G_j(x)^T G_j(x) & 0 \\ \star & \star & -I \end{bmatrix} v_2 \\
& - r_{ij}(x, v_2) \chi_{ij}(x) \in \mathcal{P}_{sos}(x, v_2) \quad\quad (41)
\end{aligned}
$$

$$v_1^T Q_i(\hat{x}^i) v_1 + p_{ij}(x, v_1) \chi_{ij}(x) = v_1^T Q_j(\hat{x}^j) v_1 \tag{42}$$

for all $(i,j) \in \tilde{I}$. Then, the Filippov solutions of the switched system $\tilde{S}$ are asymptotically stable at origin, and it holds that

$$\|y\|_{\mathcal{L}_2}^2 \le \eta^2 \|w\|_{\mathcal{L}_2}^2 + x_0^T Q_{i_0}(\hat{x}_0^i) x_0 \quad \text{for all} \quad x_0 \in \mathcal{X}$$

Proof. Equations (39) and (42) correspond to (30) and (33), respectively. Substituting A_{ci} and C_{ci} in (35) yields

$$-v^T \bigg((A_j(x) + B_j(x)K_j(x))^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i)(A_j(x) + B_j(x)K_j(x))$$

$$+ \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \big(A_j^l(x)x\big) + (C_j(x) + D_j(x)K_j(x))^T (C_j(x) + D_j(x)K_j(x))$$

$$+ \eta^{-2} Q_i(\hat{x}^i) G_j(x)^T G_j(x) Q_i(\hat{x}^i) \bigg) v < 0$$

for all $x \in bd(X_i) \cup bd(X_j)$. With some calculation, it reduces to

$$-v^T \bigg(A_j(x)^T Q_i(\hat{x}^i) + Q_i(\hat{x}^i) A_j(x) + K_j(x)^T B_j(x)^T Q_i(\hat{x}^i)$$

$$+ Q_i(\hat{x}^i) B_j(x) K_j(x) + \sum_{l \in L_j} \frac{\partial Q_i(\hat{x}^i)}{\partial x_l} \big(A_j^l(x)x\big) + C_j(x)^T C_j(x)$$

$$+ C_j(x)^T D_j(x)^T K_j(x) + K_j(x)^T D_j(x)^T C_j(x)$$

$$+ \eta^{-2} Q_i(\hat{x}^i) G_j(x)^T G_j(x) Q_i(\hat{x}^i) + K_j(x)^T D_j(x)^T D_j(x) K_j(x) \bigg) v < 0$$

which is equivalent to

$$-v^T \bigg(\Gamma_{ij}(x) + \eta^{-2} Q_i(\hat{x}^i) G_j(x)^T G_j(x) Q_i(\hat{x}^i) + K_j(x)^T D_j(x)^T D_j(x) K_j(x) \bigg) v < 0$$

Exploiting Shur's lemma, we obtain

$$-v^T \begin{bmatrix} \Gamma_{ij}(x) & Q_i(\hat{x}^i) & K_j(x)^T D_j(x)^T \\ \star & \eta^2 \big(G_j(x)^T G_j(x)\big)^{-1} & 0 \\ \star & \star & I \end{bmatrix} v < 0$$

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To avert infeasibility due to singularity of $G_j(x)^T G_j(x)$ for some x, we use the following congruence transformation

$$\begin{bmatrix} I & 0 & 0 \\ 0 & G_j(x)^T G_j(x) & 0 \\ 0 & 0 & I \end{bmatrix} \times \begin{bmatrix} \Gamma_{ij}(x) & Q_i(\hat{x}^i) & K_j(x)^T D_j(x)^T \\ \star & \eta^2\big(G_j(x)^T G_j(x)\big)^{-1} & 0 \\ \star & \star & -I \end{bmatrix}$$

$$\times \begin{bmatrix} I & 0 & 0 \\ 0 & G_j(x)^T G_j(x) & 0 \\ 0 & 0 & I \end{bmatrix}^T$$

$$= \begin{bmatrix} \Gamma_{ij}(x) & Q_i(\hat{x}^i)G_j(x)^T G_j(x) & K_j(x)^T D_j(x)^T \\ \star & -\eta^2 G_j(x)^T G_j(x) & 0 \\ \star & \star & -I \end{bmatrix}$$

Therefore, if

$$v^T \begin{bmatrix} \Gamma_{ij}(x) & Q_i(\hat{x}^i)G_j(x)^T G_j(x) & K_j(x)^T D_j(x)^T \\ \star & -\eta^2 G_j(x)^T G_j(x) & 0 \\ \star & \star & -I \end{bmatrix} v < 0$$

holds for all $x \in bd(X_i) \cup bd(X_j)$ and all $(i,j) \in \tilde{I}$, (32) is satisfied as well. In a similar fashion, it can be demonstrated that (40) is analogous to (31). This completes the proof. □

Remark 2. It is observed that the SOS feasibility tests (40) and (41) in Corollary 2 are bilinear in the variables $\{Q_i(\hat{x}^i)\}_{i \in I}$ and $\{K_i(x)\}_{i \in I}$. Bilinear SOS programs have been previously introduced by Packard and collaborators [34–36] in analyzing the region of attraction of nonlinear systems.

The following iterative algorithm is proposed to surmount the aforementioned bilinear terms in Corollary 2.

- **Initialization**: Preset $\{\epsilon_i\}_{i\in I}$, $\{q_{ik}(x, v_1)\}_{k\in N_i} \subset \mathcal{P}_{sos}(x, v_1)$, $\{w_{ik}(x, v_2)\}_{k\in N_i} \subset \mathcal{P}_{sos}(x, v_2)$, $\varepsilon_i(x, v_1) \in \mathcal{P}(x, v_1)$, $\rho_i(x, v_2) \in \mathcal{P}(x, v_2)$, $p_{ij}(x, v_1) \in \mathcal{P}(x, v_1)$, and $r_{ij}(x, v_2) \in \mathcal{P}(x, v_2)$ with $i \in I$ and $(i, j) \in \tilde{I}$. Set $iter = 0$ and predetermine $\delta > 0$ a small positive number. Select $\{K_i^0\}_{i\in I}$ as controller gains for the linearized model using a linear control design method e.g. pole placement.

- **Step Q**: Given the set of controller gains $\{K_i^{iter}(x)\}_{i\in I}$. Solve the following SOS optimization problem

$$\min_{\{Q_i(\hat{x}^i)\}_{i\in I}} \eta$$
$$\text{subject to} \quad (39) - (42)$$

  Then, set $\{Q_i^{iter}\}_{i\in I} = \{Q_i^*\}_{i\in I}$ (the superscript $*$ signifies the achieved values from the optimization problem).

- **Step K**: Given the set of symmetric polynomial matrices $\{Q_i^{iter}(\hat{x}^i)\}_{i\in I}$. Solve the following SOS optimization problem

$$\min_{\{K_i(x)\}_{i\in I}} \eta$$
$$\text{subject to} \quad (39) - (42)$$

  Subsequently, set $\{K_i^{iter+1}(x)\}_{i\in I} = \{K_i^*(x)\}_{i\in I}$ and $\eta_{k+1} = \eta^*$.

- **Finalization**: If $|\eta^{iter+1} - \eta^{iter}| \leq \delta$, return $\{K_i^{iter+1}\}_{i\in I}$, $\{Q_i^{iter}\}_{i\in I}$, and η^{iter+1} as the solutions to the SOS problem. Otherwise, $iter = iter + 1$ and go to *Step Q*.

Notice in particular that the attained value of η from the above algorithm determines the best achievable H_∞ performance of the switched system.

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5. Simulation Results

In the preceding section, we set out sufficient conditions and an algorithm to synthesize guaranteed cost H_∞ controllers for switched systems defined on semi-algebraic sets. In what follows, we evaluate the proposed schemes through simulation analysis.

Consider a switched system defined on semi-algebraic sets $\mathcal{S}$ defined by (3)–(5) . Let $\mathcal{X} \subset \mathbb{R}^2$. Suppose $i \in I = \{1, 2\}$ and

$$X_1 = \{x \in \mathcal{X} \mid -x_2^2 + x_1^3 \geq 0\}$$

$$X_2 = \{x \in \mathcal{X} \mid x_2^2 - x_1^3 \geq 0\}$$

Thus, $\chi_{12}(x) = x_2^2 - x_1^3$ since $\chi_{12}(x) = 0$ accounts for $X_1 \cap X_2$. The corresponding system polynomial matrices are given by

$$A_1(x) = \begin{bmatrix} -2 - x_1^2 & -5 - x_2^2 \\ 6 + x_1^2 & 3 - x_2^2 \end{bmatrix}, \quad B_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_1(x) = \begin{bmatrix} 0 \\ 2 + x_2^3 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} x_1 - x_1^2 & 1 \\ 4 & 2 \end{bmatrix}, \quad B_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_2(x) = \begin{bmatrix} -6 - 2x_2 \\ 0 \end{bmatrix}$$

$$C_1(x) = C_2(x) = \begin{bmatrix} x_2 & 0 \end{bmatrix}$$

The simulations verify that the solutions of the uncontrolled system are unstable as illustrated in Fig. 2.

In order to exploit the suggested approach to design switching controllers, the following parameters are preset

$$\epsilon_1 = \epsilon_2 = 0.01, \text{ and } \delta = 0.001$$

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The linearized system can be readily computed

$$A_1^{linear} = \begin{bmatrix} -2 & -5 \\ 6 & 3 \end{bmatrix}, \quad A_2^{linear} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$$

with the same B_1 and B_2 as given above. Using the pole-placement algorithm, controller gains of

$$K_1^0 = \begin{bmatrix} -5.6 & -8 \end{bmatrix}, \quad K_2^0 = \begin{bmatrix} -9 & -8.5 \end{bmatrix}$$

were determined for pole locations of $(-3, -4)$. Moreover, from the structure of system matrices, it can be deduced that $Q_1(x_1)$ and $Q_2(x_2)$. It is worth noting that if none of the rows of $[B_i \quad G_i]$ were zero, then Q_i should be assumed as a matrix whose entries are constant numbers (not polynomials in x); otherwise, one has to face increased nonlinearity that even the suggested methodologies is of no use.

The proposed H_∞ controller synthesis algorithm as described in Section 4 was employed. Fig. 3 sketches the reduction of η with respect to number of iterations. As it can be observed from the figure, the algorithm converges at 19 iterations. The SOS conditions were investigated based on SOSTOOLS [37] toolbox in MATLAB R2010b. The algorithm took 89.2603 seconds on Intel(R) Core(TM)2 Due CPU T7500 @2.20GHz and 3.00 GB of RAM.

The solutions attained from the suggested algorithm were

$$\eta = 0.739$$

$$K_1(x) = \begin{bmatrix} 0.04 & 2.62x_2^2 \end{bmatrix}, \text{ and } K_2(x) = \begin{bmatrix} -1.75 & 52.66 - 11.49x_2 \end{bmatrix}$$

$$100Q_1(x_1) = \begin{bmatrix} 17.44 + 0.46x_1^2 & -6.67 \\ -6.67 & 5.02 \end{bmatrix}, \text{ and }$$

$$100Q_2(x_2) = \begin{bmatrix} -8.83 & -11.03 + 0.05x_2 \\ -11.03 + 0.05x_2 & 39.33 \end{bmatrix},$$

Thus, from Corollary 2, all Filippov solutions of the system are asymptotically stable and the disturbance attenuation performance (7) is satisfied. Fig. 4 portrays the solutions of the closed loop system in the absence of disturbance ($w \equiv 0$). Notice in particular that the solutions with sliding modes are also asymptotically stable.

In order to examine the disturbance mitigation performance of the proposed methodology, a number of simulations were carried out that are described next. The first disturbance signal that we employed was $w(t) = 3\sin(2\pi t)$ with $t \in [0, 10]$ (see Fig. 5). Fig. 6 illustrates the disturbance attenuation performance of the system under zero initial conditions. According to Corollary 2, for the mentioned sinusoidal disturbance and zero initial conditions, one obtains

$$\|y\|_{\mathcal{L}_2}^2 \leq (0.739)^2(45) \Rightarrow \|y\|_{\mathcal{L}_2}^2 \leq 24.5754$$

The second applied signal was a square wave with duty cycle of 50 percent, values belonging to $\{3, -1\}$, $t \in [0, 10]$, and a period of 1 (see Fig. 7). Fig. 8 portrays the disturbance attenuation performance of the system under zero

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initial conditions. From Corollary 2, for the latter square wave disturbance and with zero initial conditions, it holds that

$$\|y\|_{\mathcal{L}_2}^2 \leq (0.739)^2(50) \Rightarrow \|y\|_{\mathcal{L}_2}^2 \leq 27.3061$$

The last disturbance signal that was exploited to essay the functionality of the achieved controllers is a white Gaussian noise $\mathcal{N}(0, 9)$ defined on $t \in [0, 10]$ as depicted in Fig. 9. Fig. 10 demonstrates the disturbance mitigation performance of the switched system under zero initial conditions. Interestingly, the standard deviation of the output signal is approximately 0.007 which is considerably less than that of the input disturbance. Furthermore, the same set of simulations were launched for non-zero initial conditions. These results are given in Figs. 11-13. Once again, Corollary 2 implies that

$$\|y\|_{\mathcal{L}_2}^2 \leq (0.739)^2(45) + 4.1222 \Rightarrow \|y\|_{\mathcal{L}_2}^2 \leq 28.6979$$

for the sinusoidal disturbance and initial conditions of $x_0 = (1, -3)$, and

$$\|y\|_{\mathcal{L}_2}^2 \leq (0.739)^2(50) + 0.5246 \Rightarrow \|y\|_{\mathcal{L}_2}^2 \leq 27.8307$$

for the square wave disturbance and initial conditions of $x_0 = (-1, 1)$. It is also worth mentioning that the control signals are discontinuous when the states of the switched system are in the vicinity of an attractive boundary. Taken together, these results confirm the asymptotic stability and the guaranteed cost H_∞ disturbance attenuation performance of the closed loop system.

6. Conclusions

In this paper, the aim was to develop a method to determine guaranteed cost H_∞ controllers for switched systems defined on semi-algebraic sets in the

context of Filippov solutions. The results given in this study was formulated in terms of SOS programming which can be efficiently applied based on available computational tools. Further research might explore other control performance criterions such as H_2 or mixed H_2–H_∞. Another issue that was not addressed in this study was the influence of uncertainty in the switched system dynamics when the H_∞ controller is applied. In particular, semi-algebraic uncertainty seems to be an appropriate choice in this setting.

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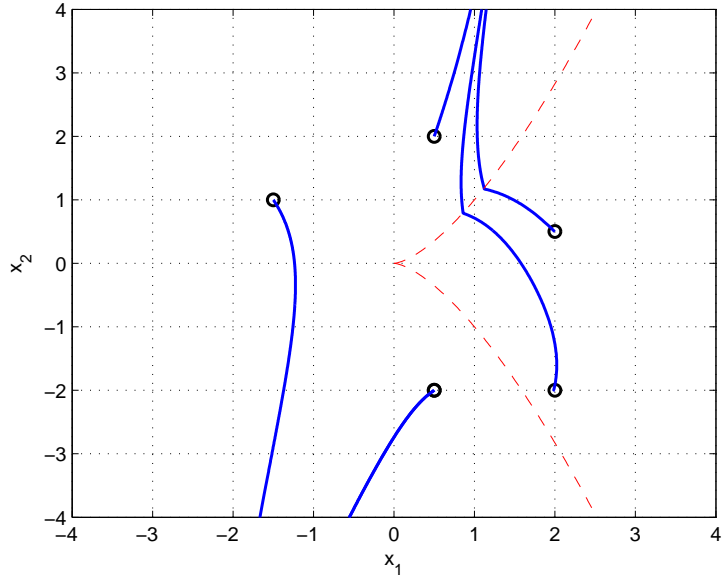
Figure 2: The trajectories of the uncontrolled switched system. The dashed lines illustrate the boundaries.
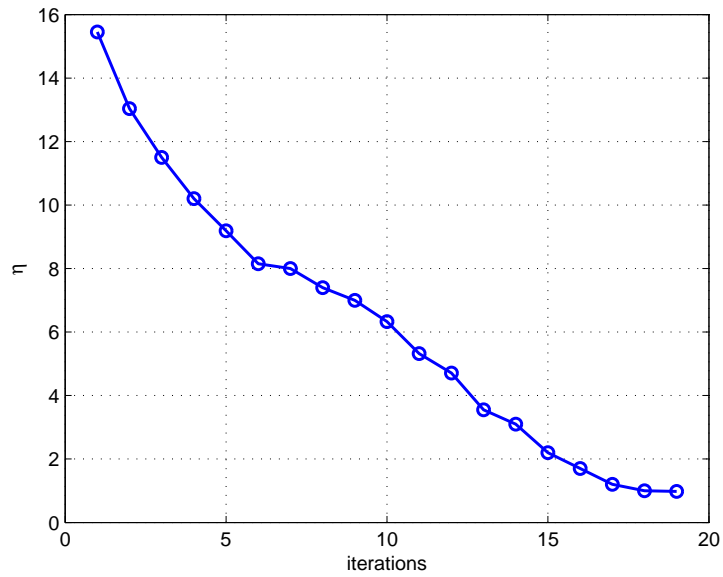
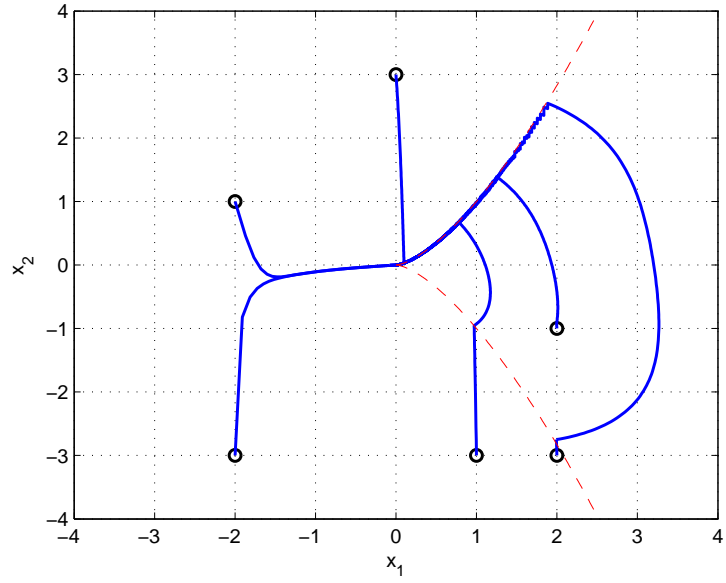Figure 3: The reduction of η as the algorithm proceeds.

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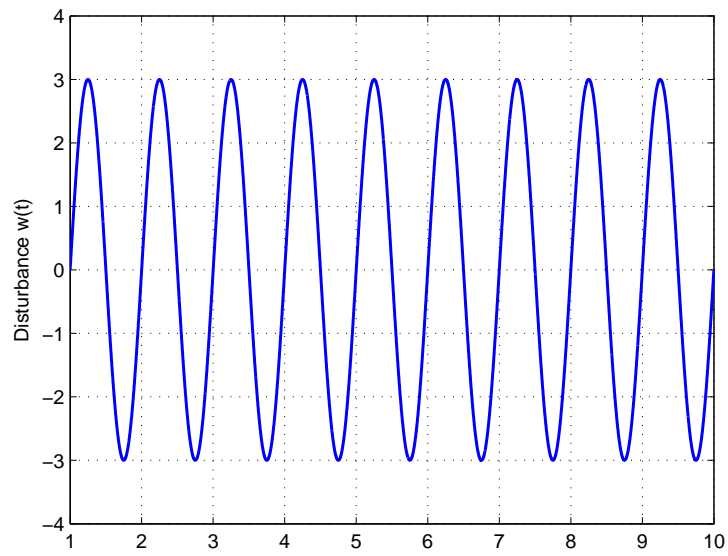Figure 4: The trajectories of the closed loop switched system.
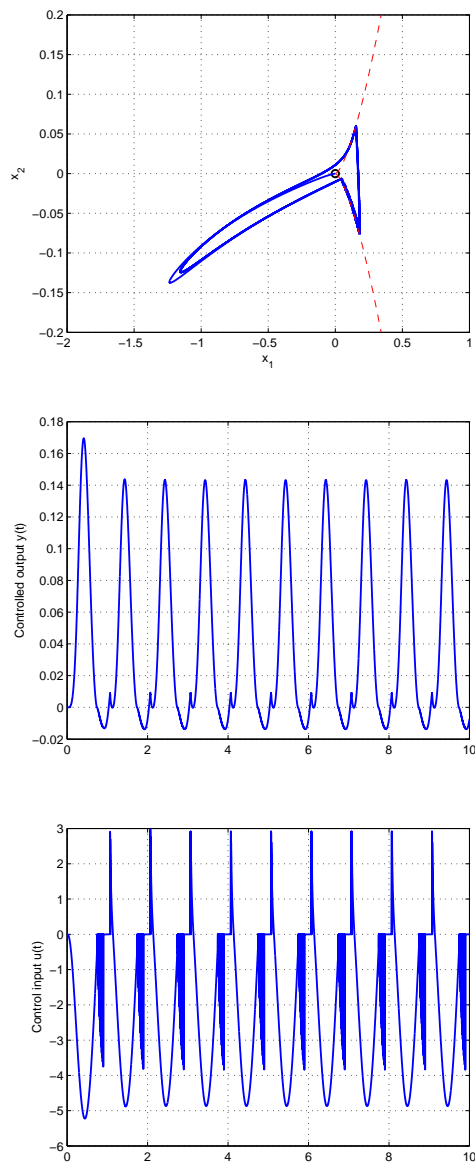

Figure 5: The applied sinusoidal disturbance.

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Figure 6: The responses of the closed loop system to disturbance under zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).
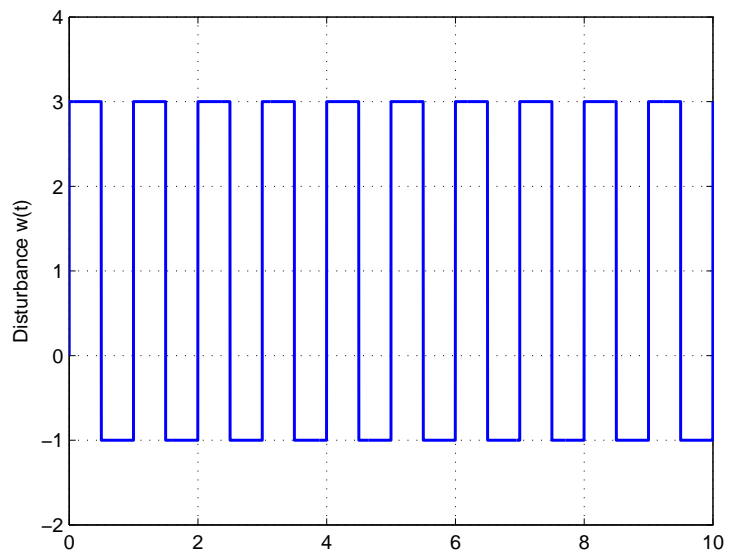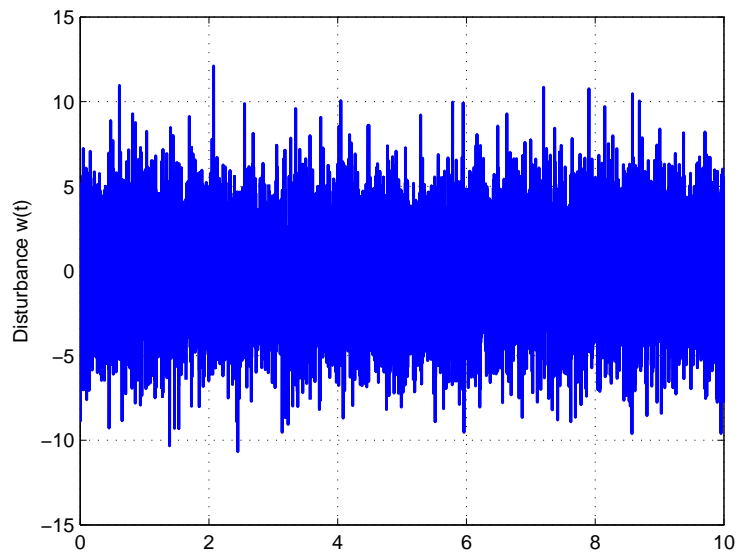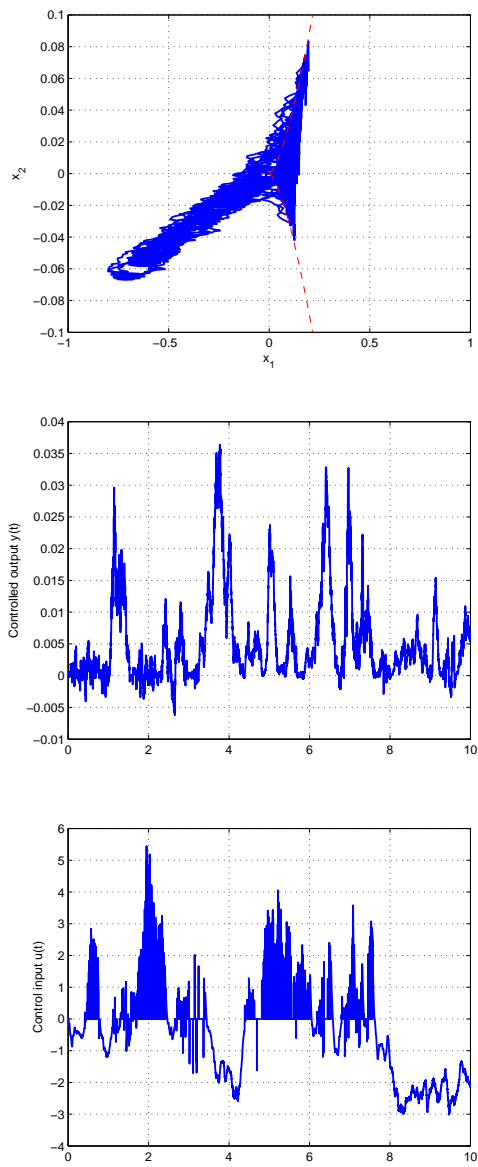
Figure 7: The applied square wave disturbance.
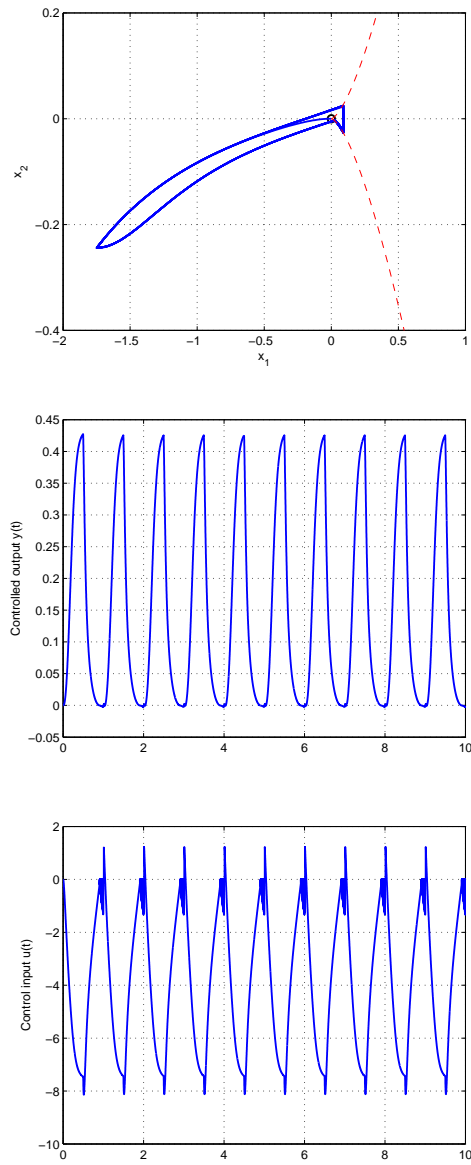
Figure 8: The responses of the closed loop system to disturbance under zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).
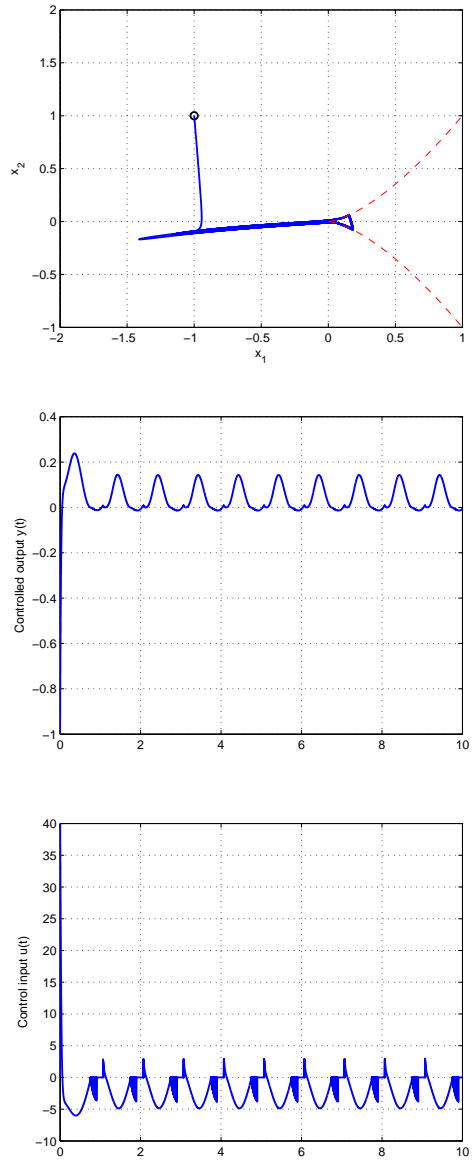
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Figure 9: The applied Gaussian random disturbance.

Figure 10: The responses of the closed loop system to disturbance under zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).

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Figure 11: The responses of the closed loop system to disturbance with non-zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).
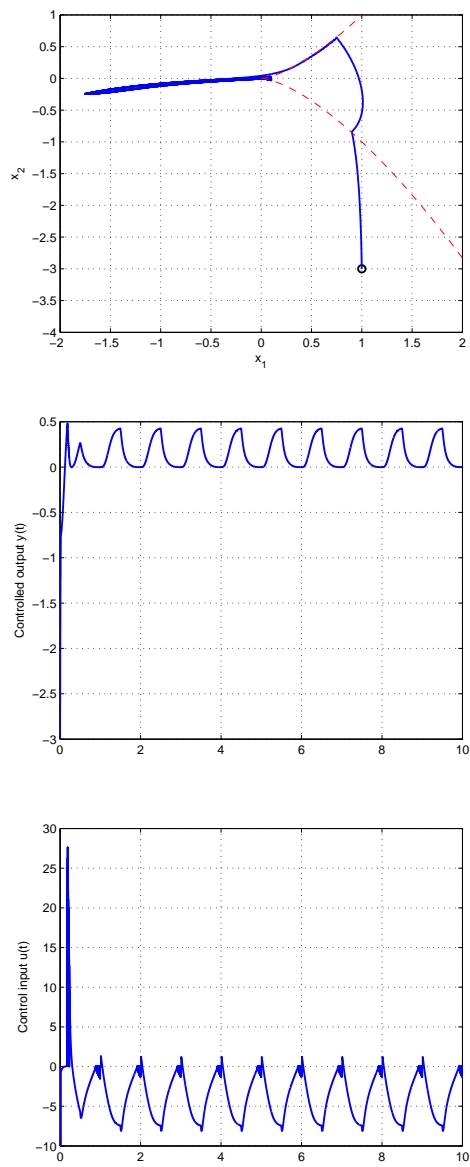
Figure 12: The responses of the closed loop system to disturbance with non-zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).
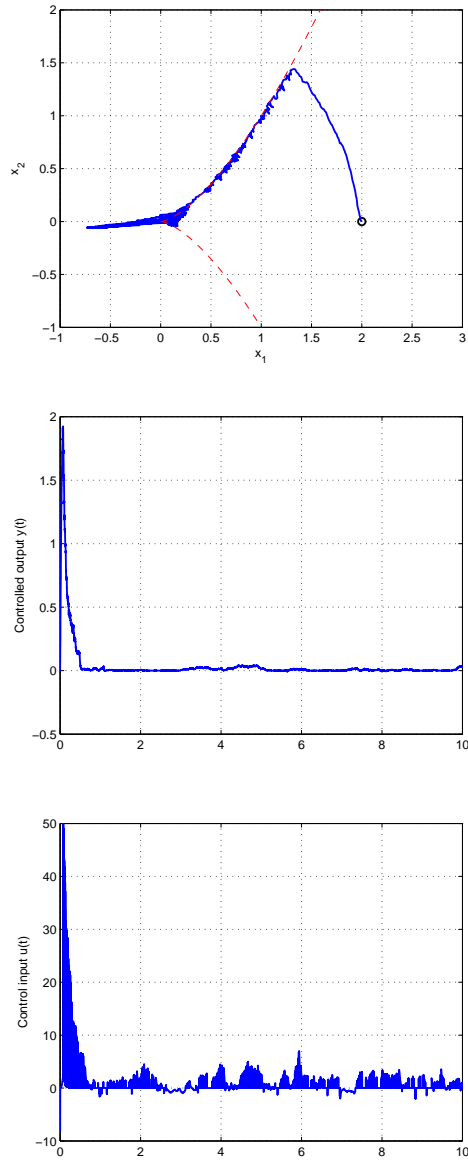
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Figure 13: The responses of the closed loop system to disturbance with non-zero initial conditions. Phase portrait (top), output signal (middle), control input (bottom).