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Input-Output Analysis of Distributed Parameter Systems Using Convex Optimization

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***Abstract*— This paper investigates input-output properties of systems described by partial differential equations (PDEs). Analogous to systems described by ordinary differential equations (ODEs), dissipation inequalities are used to establish input-output properties for PDE systems. Dissipation inequalities pertaining to passivity, induced $\mathcal{L}^2$-norm, reachability, and input-to-state stability (ISS) are formulated. For PDE systems with polynomial data, the dissipation inequalities are solved via polynomial optimization. The results are illustrated with an example.**

I. INTRODUCTION

In distributed parameter systems (PDE systems), the state is a function of both space and time. Numerous phenomena in the real world are modeled by PDE systems (see [1], [2], [3] for examples from Magnetohydrodynamics, elastic beams, fluid flows, and Tokamaks). Despite the modeling capacity of PDEs and abundant applications, they possess several analytical challenges pertaining to the fact that solutions of PDE systems belong to infinite-dimensional (function) spaces. For instance, the stability of trajectories [4] and input-output properties for a PDE system depends on the norm one considers.

Dissipation inequalities characterize different input-output properties of dynamical systems in terms of suitable storage functions/functionals and supply rates [5]. One major virtue of the dissipation inequalities is that, whenever elements of interconnections of dynamical systems are characterized by dissipation inequalities, properties of the interconnection can be obtained [6]. Solutions to dissipation inequalities were obtained with convex optimization in the context of linear ODE systems in [7] and, more recently, for nonlinear ODE systems with polynomial vector fields [8].

In the context of PDE systems, numerical solutions to dissipation inequalities have been proposed only recently. In [9], for linear time-varying hyperbolic PDE systems, an ISS dissipation inequality was proposed and the weighted $\mathcal{L}^2$ functional was considered as a storage functional. ISS Lyapunov functionals were proposed in [10] for a class of nonlinear parabolic partial differential equations. ISS has also been studied in [11] with a focus on semi-linear parabolic PDEs, and in [12] for semi-linear diffusion equation, where the concept of ISS-Lyapunov functional, given by weighted $\mathcal{L}^2$-norm, was introduced and used to formulate ISS estimates for a model of magnetic flux profile in Tokamak plasma. More general ISS definitions were presented in [13], wherein a small gain theorem for interconnection of the PDE systems was formulated.

Once the formulation of dissipation inequalities in terms of storage functionals and given supply rates is obtained, their solution (the computation of storage functionals) is, in general, difficult. Therefore, most of the studies carried out so far are based on *ad hoc* construction methods. However, numerical solutions can be expected for certain classes of storage functionals. In order to make the challenging computation of storage functionals more general (for the class of PDEs with polynomial data), we take advantage of the results in [14], where a methodology for solving integral inequalities is proposed.

The objective of this paper is to provide a framework for using dissipation inequalities for a class of PDE systems and to propose algorithmic construction methods based on convex optimization. We consider different input-output properties in the $\mathcal{L}^2$-norm. To this end, we formulate the dissipation inequalities for passivity, induced $\mathcal{L}^2$-norm boundedness, reachability and ISS. The virtue of the proposed method is that, given a PDE with polynomial data, one is able to use semi-definite programming in terms of sum-of-squares (SOS) programs to construct certificates for different properties. An example is used to illustrate the results.

The paper is organized as follows. The next section presents the notation and the problem formulation. In Section III, the dissipation inequalities for PDE systems are formulated. Section IV discusses the construction method based on semi-definite programming. An example is given in Section V to illustrate the proposed methods. Finally, Section VI concludes the paper and provides directions for future research.

II. PRELIMINARIES

A. Notation

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The space of $n \times n$ symmetric real matrices is denoted by $\mathbb{S}^n$. The identity matrix is denoted by I. A domain $\Omega \subset \mathbb{R}^n$ is a connected, open subset of $\mathbb{R}^n$, and $\overline{\Omega}$ is the closure of set Ω. The boundary $\partial\Omega$ of set Ω is defined as $\overline{\Omega} \setminus \Omega$ with $\setminus$ denoting set subtraction. The space of p-th power integrable functions defined over Ω is denoted $\mathcal{L}^p_\Omega$ endowed with the norm

$$\|(\cdot)\|_{\mathcal{L}^p_\Omega} = \left(\int_\Omega (\cdot)^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

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for $1 \leq p < \infty$, and

$$\|(\cdot)\|_{\mathcal{L}^{\infty}_{\Omega}} = \sup_{x\in\Omega} |(\cdot)|,$$

for $p = \infty$. Also, we denote by $\mathcal{L}^2_{\Omega,[t_0,T]}$, with $t_0 \geq 0$, the space of square integrable functions in $x \in \Omega$ and $t \in [t_0, T]$ with the norm

$$\|(\cdot)\|_{\mathcal{L}^2_{\Omega,[t_0,T)}} = \left(\int_{\Omega} \int_{t_0}^{T} (\cdot)^T (\cdot) \, \mathrm{d}t\mathrm{d}x \right)^{\frac{1}{2}}.$$

The space of k-times continuous differentiable functions defined on Ω is denoted by $\mathcal{C}^k(\Omega)$. If $p \in \mathcal{C}^1$, then $\partial_x p$ is used to denote the derivative of p with respect to variable x, i.e. $\partial_x := \frac{\partial}{\partial x}$. In addition, we adopt Schwartz's multi-index notation. For $u \in \mathcal{C}^{k,n}$, $\alpha \in \mathbb{N}^n_0$, define

$$D^{\alpha} u := \left(u_1, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial^{\alpha_1} u_1}{\partial x^{\alpha_1}}, \ldots, u_n, \frac{\partial u_n}{\partial x}, \ldots, \frac{\partial^{\alpha_n} u_n}{\partial x^{\alpha_n}} \right).$$

A continuous strictly increasing function $k : [0, a) \to \mathbb{R}_{\geq 0}$, satisfying $k(0) = 0$, belongs to class $\mathcal{K}$. If $a = \infty$ and $\lim_{x\to\infty} k(x) = \infty$, k belongs to class $\mathcal{K}_{\infty}$. We recall that, for any class $\mathcal{K}$ function, the inverse exists and belongs to $\mathcal{K}$. Furthermore, for any positive a and b and class $\mathcal{K}$ function k, we have [15, Inequality (12)]

$$k(a + b) \leq k(2a) + k(2b).$$

For a symmetric matrix function $S(x)$, we define $\underline{\lambda}(S) = \inf_{x\in\Omega} |\lambda_{min}(S(x))|$, where $\lambda_{min} : \mathbb{S}^n \to \mathbb{R}$ is the minimum eigenvalue function. Similarly, $\bar{\lambda}(S) = \sup_{x\in\Omega} |\lambda_{max}(S(x))|$, where $\lambda_{max} : \mathbb{S}^n \to \mathbb{R}$ is the maximum eigenvalue function.

B. Problem Formulation

We consider the class of PDE systems described by

$$\begin{aligned} \partial_t u &= F(x, D^{\alpha_u} u, D^{\alpha_d} d), \\ y &= H(x, D^{\delta} u), \quad \forall x \in \Omega, \ t > t_0 \end{aligned} \tag{1}$$

where, $u \in \mathcal{L}^{2,n}_{\Omega}$, $d \in \mathcal{L}^{2,m}_{\Omega}$, and $y \in \mathcal{L}^{2,q}_{\Omega}$ are dependent variables (defined over both space and time) representing states, inputs, and outputs, respectively. Independent variables t and x represent time and space, respectively. It is assumed that PDE (1) is well-posed; i.e., a solution to (1) exists and is unique.

Definition 1: An equilibrium $\psi(x)$ of (1), satisfying $F(x, D^{\alpha_u}\psi, 0) = 0$, is

- stable in $\mathcal{L}^2_{\Omega}$, if for any $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that for all $t \geq t_0$
$$\|u(x, 0) - \psi(x)\|_{\mathcal{L}^2_{\Omega}} < \delta \Rightarrow \|u(x, t) - \psi(x)\|_{\mathcal{L}^2_{\Omega}} < \varepsilon,$$
- asymptotically stable in $\mathcal{L}^2_{\Omega}$, if it is stable and $\exists \delta > 0$ such that
$$\|u(x, 0)-\psi(x)\|_{\mathcal{L}^2_{\Omega}} < \delta \Rightarrow \lim_{t\to\infty} \|u(x, t)-\psi(x)\|_{\mathcal{L}^2_{\Omega}} = 0,$$
- exponentially stable in $\mathcal{L}^2_{\Omega}$, if there exists a $\lambda > 0$, such that for all $t \geq t_0$
$$\|u(x, t) - \psi(x)\|^2_{\mathcal{L}^2_{\Omega}} \leq \|u(x, 0) - \psi(x)\|^2_{\mathcal{L}^2_{\Omega}} e^{-\lambda(t-t_0)}.$$

In the sequel, we consider stability to the null solution, i.e. $\psi(x) = 0, \ \forall x \in \Omega$ in Definition 1.

In order to study input-output properties of system (1), we define properties as follows.

Definition 2 (input-output Properties):

A. *Passivity*: System (1) satisfies the following inequality
$$0 \leq \int_{t_0}^{\infty} \int_{\Omega} d^T(x, t) y(x, t) \, \mathrm{d}x\mathrm{d}t, \tag{2}$$
subject to $u(x, t_0) = 0, \ \forall x \in \Omega$.

B. *Reachability*: The solutions of (1) satisfy
$$\|u(x, T)\|_{\mathcal{L}^2_{\Omega}} \leq \beta \left(\|d(x, t)\|_{\mathcal{L}^2_{\Omega,[t_0,T)}} \right), \quad \forall T > 0 \tag{3}$$
for $\beta \in \mathcal{K}_{\infty}$ and subject to $u(x, t_0) = 0, \ \forall x \in \Omega$.

C. *Induced $\mathcal{L}^2$-norm Boundedness*: For some $\gamma > 0$,
$$\|y(x, t)\|_{\mathcal{L}^2_{\Omega,[t_0,\infty)}} \leq \gamma \|d(x, t)\|_{\mathcal{L}^2_{\Omega,[t_0,\infty)}} \tag{4}$$
subject to zero initial conditions $u(x, t_0) = 0, \ \forall x \in \Omega$.

D. *Input-to-State Stability*: For some scalar $\psi > 0$, functions $\beta, \tilde{\beta}, \chi \in \mathcal{K}_{\infty}$, and $\sigma \in \mathcal{K}$, it holds that
$$\begin{aligned} \|u(x, t)\|_{\mathcal{L}^2_{\Omega}} \leq{}& \beta\Big(e^{-\psi(t-t_0)} \chi \left(\|u(x, t_0)\|_{\mathcal{L}^2_{\Omega}} \right) \Big) \\ &+ \tilde{\beta}\Big(\sup_{\tau\in[t_0,t)} \big(\int_{\Omega} \sigma\big(|d(x, \tau)|\big) \, \mathrm{d}x \big) \Big), \end{aligned} \tag{5}$$
for all $t > t_0$.

Remark 1: The above definition of passivity is taken from [16], where a passivity-based design strategy for flow control is presented. □

Remark 2: In item C in Definition 2, for PDE system (1), we are interested in estimating upper bounds on $\gamma^* > 0$ defined as

$$\gamma^* = \sup_{0<\|d\|_{\mathcal{L}^2_{\Omega,[t_0,\infty)}}<\infty} \frac{\|y\|_{\mathcal{L}^2_{\Omega,[t_0,\infty)}}}{\|d\|_{\mathcal{L}^2_{\Omega,[t_0,\infty)}}}, \tag{6}$$

which is the induced $\mathcal{L}^2_{\Omega}$-norm of the system. □

Remark 3: It is worth noting that the ISS property (5) assures exponential convergence to the null solution in $\mathcal{L}^2_{\Omega}$ when $d \equiv 0$. Moreover, as $t \to \infty$, the first term on the right-hand side of (5) vanishes yielding

$$\begin{aligned} \lim_{t\to\infty} \|u(x, t)\|_{\mathcal{L}^2_{\Omega}} &\leq \beta \left(\int_{\Omega} \|\sigma(|d(x, t)|)\|_{\mathcal{L}^{\infty}_{[t_0,\infty)}} \, \mathrm{d}x \right) \\ &\leq \beta \left(\int_{\Omega} \sigma(\|d(x, t)\|_{\mathcal{L}^{\infty}_{[t_0,\infty)}}) \, \mathrm{d}x \right), \end{aligned} \tag{7}$$

wherein, the fact that σ and $\beta \in \mathcal{K}$ is used. Hence, when the input is bounded in $\mathcal{L}^{\infty}_{[t_0,\infty)}$ (this encompasses persistent inputs), the *state* u is bounded in $\mathcal{L}^2_{\Omega}$ sense. This is analogous to the ISS property for ODE systems [17]. □

In the sequel, we use the concept of *zero-state detectability* for PDE systems, which is defined next (for the case of ODE systems refer to [18, p. 362]).

Definition 3: A system is zero-state detectable (ZSD), if $y = 0$ implies $u = 0$.

Remark 4: Zero-state detectability imposes constraints on H in (1) ($H(x, D^\delta u) = 0 \Rightarrow u = 0$). In the special case of $H(x, D^\delta u) = h(x)u$, this is equivalent to $\nexists x \in \Omega$ such that $h(x) = 0$, thereby $y = 0 \Rightarrow u = 0$. □

In the following section, we derive conditions in terms of dissipation inequalities for properties A-D to hold.

III. Dissipation Inequalities for PDEs

In the next theorem, we formulate the dissipation inequalities associated with properties A-D in Definition 2.

Theorem 1: Consider the PDE system described by (1). If there exist a positive semidefinite storage functional $S(u)$, scalars $\gamma, \psi > 0$, and functions $\beta_1, \beta_2 \in \mathcal{K}_\infty$, $\alpha, \sigma \in \mathcal{K}$ satisfying $\psi|U| \leq \alpha(|U|)$, such that

I)

$$\partial_t S(u) \leq \int_\Omega d^T(x,t) y(x,t) \, \mathrm{d}x, \tag{8}$$

II)

$$\beta_1(\|u(x,t)\|_{\mathcal{L}^2_\Omega}) \leq S(u), \tag{9}$$

$$\partial_t S(u) \leq \gamma^2 \int_\Omega d^T(x,t) d(x,t) \, \mathrm{d}x, \tag{10}$$

III)

$$y \equiv 0 \Rightarrow u \equiv 0, \tag{11}$$

$$\begin{aligned} \partial_t S(u) \quad \leq \quad & -\int_\Omega y^T(x,t) y(x,t) \, \mathrm{d}x \\ & + \gamma^2 \int_\Omega d^T(x,t) d(x,t) \, \mathrm{d}x, \end{aligned} \tag{12}$$

IV)

$$\beta_1(\|u(x,t)\|_{\mathcal{L}^2_\Omega}) \leq S(u) \leq \beta_2(\|u(x,t)\|_{\mathcal{L}^2_\Omega}), \tag{13}$$

$$\partial_t S(u) \leq -\alpha(S(u)) + \int_\Omega \sigma(|d(x,t)|) \, \mathrm{d}x, \tag{14}$$

for all $t > t_0$, then, respectively, system (1)

I) satisfies the *passivity* property (2),

II) satisfies the *reachability* property (3) with $\beta(\cdot) = \beta_1^{-1}(\gamma(\cdot))$,

III) is asymptotically stable and its induced $\mathcal{L}^2$*-norm* is bounded by γ in (4).

IV) is *ISS* and satisfies (5) with $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1} \circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1} \circ \frac{2}{\psi}(\cdot)$.

Proof: Each item is proven as follows:

I) Integrating both sides of (8) over time from t_0 to ∞ yields

$$\int_{t_0}^\infty \partial_t S(u) \, \mathrm{d}t \leq \int_{t_0}^\infty \int_\Omega d^T y \, \mathrm{d}x\mathrm{d}t.$$

That is,

$$\lim_{t \to \infty} S\left(u(x,t)\right) - S\left(u(x,t_0)\right) \leq \int_{t_0}^\infty \int_\Omega d^T y \, \mathrm{d}x\mathrm{d}t.$$

By hypothesis, $S(u)$ is positive semidefinite. Hence, for $u(x,t_0) = 0$, we have $S(u(x,t_0)) = 0$. Moreover, $\lim_{t\to\infty} S\left(u(x,t)\right)) \geq 0$. Therefore, we obtain the passivity estimate (2).

II) Integrating both sides of (10) over time from t_0 to T yields

$$\int_{t_0}^T \partial_t S(u) \leq \gamma \int_{t_0}^T \int_\Omega d^T(x,t) d(x,t) \, \mathrm{d}x\mathrm{d}t.$$

That is,

$$S(u(x,T)) - S(u(x,t_0)) \leq \gamma \|d\|_{\mathcal{L}^2_{\Omega,[t_0,T)}}.$$

Noting that, with $u(x,t_0) = 0$, from (9), we have

$$\beta_1(\|u(x,T)\|_{\mathcal{L}^2_\Omega}) \leq S(u(x,T)) \leq \gamma \|d\|_{\mathcal{L}^2_{\Omega,[t_0,T)}}.$$

Since $\beta_1 \in \mathcal{K}_\infty$, its inverse exists and belongs to $\mathcal{K}_\infty$. Thus,

$$\|u(x,T)\|_{\mathcal{L}^2_\Omega} \leq \beta_1^{-1}\left(\gamma \|d\|_{\mathcal{L}^2_{\Omega,[t_0,T)}}\right).$$

Therefore, an estimate of the reachable set in term of $\|d\|_{\mathcal{L}^2_{\Omega,[t_0,T)}}$ is attained.

III) Subject to zero inputs $d \equiv 0$, (12) becomes

$$\partial_t S(u) \leq -\int_\Omega y^T y \, \mathrm{d}x \tag{15}$$

Inequality (15) implies that the time derivative of the storage functional $S(u)$ is negative semidefinite. Moreover, condition (11) is equivalent to system (1) being ZSD. Thus, $\partial_t S(u) = 0$ only if $u = 0$. Hence, from LaSalle's invariance principle [19, Theorem 3.64, p. 161], it follows that u converges to the null solution $u = 0$ asymptotically.

Furthermore, by integrating both sides of (12) from t_0 to ∞, we obtain

$$\int_{t_0}^\infty \partial_t S(u) \, \mathrm{d}t \leq \\ -\int_{t_0}^\infty \int_\Omega y^T y \, \mathrm{d}x\mathrm{d}t + \gamma^2 \int_{t_0}^\infty \int_\Omega d^T d \, \mathrm{d}x\mathrm{d}t.$$

That is,

$$\lim_{t\to\infty} S(u(x,t)) - S(u(x,t_0)) \leq \\ -\int_{t_0}^\infty \int_\Omega y^T y \, \mathrm{d}x\mathrm{d}t + \gamma^2 \int_{t_0}^\infty \int_\Omega d^T d \, \mathrm{d}x\mathrm{d}t.$$

Since $S(\cdot)$ is positive semidefinite and $u(x,t_0) = 0, \ x \in \Omega$, we have

$$\lim_{t\to\infty} S(u(x,t)) \leq -\int_{t_0}^\infty \int_\Omega y^T y \, \mathrm{d}x\mathrm{d}t + \int_{t_0}^\infty \int_\Omega d^T d \, \mathrm{d}x\mathrm{d}t,$$

and we obtain

$$\int_{t_0}^\infty \int_\Omega y^T y \, \mathrm{d}x\mathrm{d}t \leq \gamma^2 \int_{t_0}^\infty \int_\Omega d^T d \, \mathrm{d}x\mathrm{d}t,$$

which is identical to (4).

IV) By rearranging the terms in (14) and using the assumption that $\psi|U| \leq \alpha(|U|)$, we have

$$\partial_t S(u) + \psi S(u) \leq \int_\Omega \sigma(|d|) \, \mathrm{d}x. \tag{16}$$

With the strictly increasing function $e^{\psi(t-t_0)}$, we have

$$e^{\psi(t-t_0)}\left(\partial_t S(u) + \psi S(u)\right) \le e^{\psi(t-t_0)} \int_\Omega \sigma(|d|)\ \mathrm{d}x.$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\psi(t-t_0)} S\right) \le e^{\psi(t-t_0)} \int_\Omega \sigma(|d|)\ \mathrm{d}x. \tag{17}$$

Integrating both sides of inequality (17) from t_0 to t gives

$$\begin{aligned} & e^{\psi(t-t_0)} S(u(x,t)) - S(u(x,t_0)) \\ \le\ & \int_{t_0}^{t} e^{\psi(\tau-t_0)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right) \mathrm{d}\tau \\ \le\ & \left(\int_{t_0}^{t} e^{\psi(\tau-t_0)}\ \mathrm{d}\tau\right) \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right) \\ \le\ & \frac{1}{\psi}(e^{\psi(t-t_0)} - 1) \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right) \\ \le\ & \frac{e^{\psi(t-t_0)}}{\psi} \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right), \end{aligned} \tag{18}$$

where Hölder's inequality[1] is used in the second inequality above. Dividing the terms above by the positive term $e^{\psi(t-t_0)}$ gives

$$\begin{aligned} S(u(x,t)) \ \le\ & e^{-\psi(t-t_0)} S(u(x,t_0)) \\ & + \frac{1}{\psi} \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right). \end{aligned}$$

Using (13), we infer that

$$\begin{aligned} \beta_1(\|u(x,t)\|_{\mathcal{L}^2_\Omega}) \ \le\ & e^{-\psi(t-t_0)} \beta_2(u(x,t_0)) \\ & + \frac{1}{\psi} \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right). \end{aligned}$$

Since $\beta_1 \in \mathcal{K}_\infty$, its inverse exists and belongs to $\mathcal{K}_\infty$. Hence,

$$\begin{aligned} \|u(x,t)\|_{\mathcal{L}^2_\Omega} \le\ & \beta_1^{-1}\left(2e^{-\psi(t-t_0)} \beta_2(u(x,t_0))\right) \\ & + \beta_1^{-1}\left(\frac{2}{\psi} \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right)\right). \end{aligned}$$

It suffices to let $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1} \circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1} \circ \frac{2}{\psi}(\cdot)$. ■

IV. Computation of Storage Functionals

For computational purposes, the following structure is considered as a candidate storage functional to check the dissipation inequalities given in Theorem 1

$$S(u) = \frac{1}{2} \int_\Omega u^T P(x) u\ \mathrm{d}x \tag{19}$$

[1] Let $p, q \in [1,\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all measurable functions f and g, it holds that

$$\|fg\|_{\mathcal{L}^1} \le \|f\|_{\mathcal{L}^p} \|g\|_{\mathcal{L}^q}$$

wherein, $P(x) : \Omega \to \mathbb{S}$ is a symmetric positive definite polynomial matrix function for all $x \in \Omega$. This storage functional candidate satisfies

$$\frac{1}{2}\underline{\lambda}(P)\|u\|^2_{\mathcal{L}^2_\Omega} \le S(u) \le \frac{1}{2}\bar{\lambda}(P)\|u\|^2_{\mathcal{L}^2_\Omega}. \tag{20}$$

Therefore, $S^{\frac{1}{2}}(u)$ is equivalent to the $\mathcal{L}^2$-norm.

Remark 5: From (20), it follows that (9) and (13) are satisfied, respectively, with $\beta_1(\cdot) = \frac{\underline{\lambda}(P)}{2}(\cdot)^2$, $\beta_1^{-1}(\cdot) = \sqrt{\frac{2}{\underline{\lambda}(P)}(\cdot)}$, and $\beta_2(\cdot) = \frac{\bar{\lambda}(P)}{2}(\cdot)^2$. □

Let $\eta = \gamma^2$. For reachability analysis, we solve the following minimization problem

Problem 1:

$$\begin{aligned} & \textit{minimize } \eta \\ & \textit{subject to } (10),\ \nu^2 I < P(x), \end{aligned} \tag{21}$$

where, ν is a constant. In this case, the reachability estimate (3) transforms to

$$\|u(x,T)\|_{\mathcal{L}^2_\Omega} \le \frac{\gamma}{\nu}\|d\|_{\mathcal{L}^2_{\Omega,[t_0,T)}}, \quad \forall T > 0. \tag{22}$$

Analogously, for induced $\mathcal{L}^2$-norm, the following minimization problem is solved

Problem 2:

$$\begin{aligned} & \textit{minimize } \eta \\ & \textit{subject to } (12). \end{aligned} \tag{23}$$

When adopting the storage functional structure (19) for ISS, it is possible to check the condition

$$\partial_t S(u) \le -\int_\Omega u^T \alpha(x) u\ \mathrm{d}x + \int_\Omega \sigma(|d(x,t)|)\ \mathrm{d}x,$$

instead of (14), where $\alpha : \Omega \to \mathbb{S}^n$ is a symmetric positive definite polynomial function for all $x \in \Omega$. In this case, the ISS estimate translates to

$$\begin{aligned} \|u(x,t)\|_{\mathcal{L}^2_\Omega} \le\ & \left(e^{-\frac{\underline{\lambda}(\alpha)}{\bar{\lambda}(P)}(t-t_0)} \left(\|u(x,t_0)\|^2_{\mathcal{L}^2_\Omega}\right)\right)^{\frac{1}{2}} \\ & + \left(\frac{1}{\underline{\lambda}(\alpha)} \sup_{\tau\in[t_0,t)} \left(\int_\Omega \sigma(|d(x,\tau)|)\ \mathrm{d}x\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{24}$$

A. Convex Formulation

Provided that the problem data are polynomial in the dependent variables, one can formulate convex optimization problems (SOS programs) to solve the inequalities in Theorem 1. In the following, we summarize the approach given in [14] to solve integral inequalities with polynomial integrands. We treat integral inequalities of the type

$$\int_\Omega \begin{bmatrix} D^\alpha u \\ D^\beta d \end{bmatrix}^T M(x) \begin{bmatrix} D^\alpha u \\ D^\beta d \end{bmatrix}\ \mathrm{d}x \ge 0, \tag{25}$$

where $M : \Omega \to \mathbb{S}^{d_M}$ with $d_M = \sum_{i=1}^{n}(1+\alpha_i) + \sum_{j=1}^{m}(1+\beta_j)$. The entries of the vector $\begin{bmatrix}(D^\alpha u)^T & (D^\beta d)^T\end{bmatrix}^T$ in the quadratic expression are not independent, and can be related via the Fundamental Theorem of Calculus as follows

$$
\begin{aligned}
&\begin{bmatrix} D^{\alpha-1}u(1,t) \\ D^{\beta-1}d(1,t) \end{bmatrix}^T H(1) \begin{bmatrix} D^{\alpha-1}u(1,t) \\ D^{\beta-1}d(1,t) \end{bmatrix} \\
&\quad - \begin{bmatrix} D^{\alpha-1}u(0,t) \\ D^{\beta-1}d(0,t) \end{bmatrix}^T H(0) \begin{bmatrix} D^{\alpha-1}u(0,t) \\ D^{\beta-1}d(0,t) \end{bmatrix} \\
&\quad = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}x} \left(\begin{bmatrix} D^{\alpha}u \\ D^{\beta}d \end{bmatrix}^T H(x) \begin{bmatrix} D^{\alpha}u \\ D^{\beta}d \end{bmatrix} \right) \mathrm{d}x \\
&\quad := \int_\Omega \begin{bmatrix} D^{\alpha}u \\ D^{\beta}d \end{bmatrix}^T \bar{H}(x) \begin{bmatrix} D^{\alpha}u \\ D^{\beta}d \end{bmatrix} \mathrm{d}x, \qquad (26)
\end{aligned}
$$

where $H : \Omega \to \mathbb{S}^{d_M}$.

Therefore, given a subspace of Hilbert space in which (u, d) is defined such as

$$
\mathcal{B}(B) = \left\{ (u,d) \middle| B \begin{bmatrix} D^{\alpha-1}u(1,t) \\ D^{\beta-1}d(1,t) \\ D^{\alpha-1}u(0,t) \\ D^{\beta-1}d(0,t) \end{bmatrix} = 0 \right\}, \qquad (27)
$$

for some $B^T \in \mathbb{R}^{d_B}$ with $d_B = 2\left(\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j\right)$, the next result follows.

Lemma 1: If $\exists H(x) \in \mathcal{C}^1$ such that

$$
M(x) + \bar{H}(x) \geq 0, \ \forall x \in \Omega, \qquad (28)
$$

with $M(x)$ as in (25) and $\bar{H}(x)$ as in (26) and

$$
\begin{aligned}
&\begin{bmatrix} D^{\alpha-1}u(1,t) \\ D^{\beta-1}d(1,t) \end{bmatrix}^T H(1) \begin{bmatrix} D^{\alpha-1}u(1,t) \\ D^{\beta-1}d(1,t) \end{bmatrix} \\
&\quad - \begin{bmatrix} D^{\alpha-1}u(0,t) \\ D^{\beta-1}d(0,t) \end{bmatrix}^T H(0) \begin{bmatrix} D^{\alpha-1}u(0,t) \\ D^{\beta-1}d(0,t) \end{bmatrix} \leq 0, \\
&\qquad \forall (u,d) \in \mathcal{B}(B), \qquad (29)
\end{aligned}
$$

then inequality (25) holds for all $(u, d) \in \mathcal{B}(B)$.

Proof: See [14]. ∎

Testing whether (28) holds can be performed by semi-definite programming [20], [21]. In this regard, Putinar's Positivstellensatz [22, Theorem 2.14] is used to formulate the associated polynomial positivity tests that can be handled by semi-definite programming.

Remark 6: The advantage of the above method is that, once the dissipation inequalities are formulated with polynomial integrands, we can use convex optimization to check the inequalities and construct the certificates. □

V. Numerical Example

This section illustrates the application of the dissipation inequalities to a PDE system given by the heat equation with a reaction term.

A. Example : Heat Equation with Reaction Term

Consider the following PDE system

$$
\begin{aligned}
\partial_t u &= \partial_x^2 u + \lambda(x)u + \epsilon(x)d, \quad \forall x \in (0,1) \text{ and } \forall t \geq 0 \\
y &= u, \qquad (30)
\end{aligned}
$$

subject to $u(0,t) = u(1,t) = 0$ for all $t \geq 0$. For $d = 0$, the system is exponentially stable for $\lambda(x) = \lambda_0 < \pi^2$ [23, p. 11]. For passivity analysis, let $\epsilon(x) = 1$ and $\lambda(x) = \lambda_0$. Applying condition (8) in Theorem 1, certificates could be found that passivity property holds for $\lambda_0 < 0.2\pi^2$. Therefore, the upper bound of a constant coefficient λ_0 for which certificates of passivity could be found is smaller than the bound for stability.

In case of reachability analysis, let $\epsilon(x) = 100x(1-x)$ and $\lambda = 0$. With this choice of the function $\epsilon(x)$, the uniform (over the domain) input d has its maximum amplification at $x = 0.5$. The polynomial $P(x)$ is set to 1, so that the Lyapunov functional represents the $\mathcal{L}^2_\Omega$-norm of solutions. Table I provides the attained results. As expected, for larger values of parameter λ, as the system approaches the instability bound, the reachable set is enlarged.

In case of induced $\mathcal{L}^2$-norm, certificates could not be found for $\mathcal{L}^2$-norm boundedness for $\lambda(x) = \lambda_0 > 0.4\pi^2$ independently of the degree of the involved polynomials. Table II presents the results from the numerical experiments. It can be deduced from the table that, from $\lambda_0 = 0.3\pi^2$ to $\lambda_0 = 0.4\pi^2$, the induced $\mathcal{L}^2$-gain increases. At this point, take $\lambda(x) = \lambda_c - 24x + 24x^2$ and $\epsilon(x) = 100x(1-x)$. Figure 1 depicts the spatially varying parameter $\lambda(x)$ with different values of λ_c. As it can be observed, for $\lambda_c \in \{10, 11, 12, 13\}$, the coefficients exceed the stability bound for λ when it is constant; i.e., $\lambda(x) = \lambda_0 = \pi^2$. Table III summarizes the obtained results.

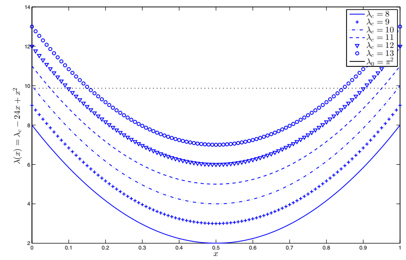

Fig. 1: The spatially varying coefficients for Equation (30).

TABLE I: Reachability analysis results for Equation (30).

$\frac{\lambda}{\pi^2}$	0	0.2	0.4	0.6	0.8
γ	5.76	6.79	8.62	12.46	29.71

TABLE II: Induced $\mathcal{L}^2$-norm results for Equation (30) subject to constant coefficients.

$\frac{\lambda}{\pi^2}$	0	0.2	0.3	0.35	0.39
γ^2	0.0560	0.1876	0.5465	1.296	6.158
Total Time (s)	16.87	18.09	18.35	16.89	18.23

TABLE III: Induced $\mathcal{L}^2$-norm results for Equation (30) subject to spatially varying coefficients.

λ_c	8	9	10	11	12	13
γ^2	6.503	6.987	4.612	5.989	7.676	10.261
Total Time (s)	16.87	18.09	18.35	16.89	18.23	

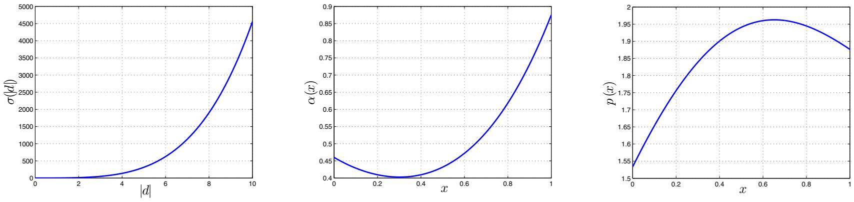

Fig. 2: ISS certificates for Equation (30) (with $\lambda = 0.2\pi^2$).

Finally, certificates for ISS are studied. The experiments were performed with construction of polynomials $P(x)$, $\alpha(x)$, and $\sigma(u)$ to certify ISS property. It turned out that certificates for ISS property could be constructed for $\lambda(x) = \lambda_0 \leq 0.5\pi^2$. Fig. 2 presents the results obtained from numerical experiments for $\lambda_0 = 0.2\pi^2$.

VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

We proposed a method to characterize the input-output properties of PDE systems with in-domain inputs and outputs, by formulating the problem using a set of dissipation inequalities. For PDEs with polynomial data in dependent and independent variables, certificates for input-output properties have been constructed by convex optimization. An example illustrated the results.

B. Future Work

In several scenarios, one is interested in evaluating a system property, which may require the cumbersome task of computing the numerical solution. For instance, in applications in fluid mechanics, it is desirable to study the effect of perturbations on the forces acting on a body and computing the solutions may be highly computationally demanding. We believe that the use of dissipation inequalities may help to eliminate the need for computing the solutions. In this study, we considered in-domain perturbations. One can formulate input-output bounds when the inputs are applied to the PDE system through boundaries. Moreover, interconnections and derivation of small-gain theorems and passivity properties are currently under study. In particular, we are interested in the interconnections of ODE and PDE systems with interconnection at the boundaries. Another interesting problem currently under study is to compute bounds on output functionals of PDE systems using convex optimization.

REFERENCES

[1] A. Gahlawat, E. Witrant, M. Peet, and M. Alamir, "Bootstrap current optimization in Tokamaks using sum-of-squares polynomials," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, Dec 2012, pp. 4359–4365.

[2] F. Castillo, E. Witrant, C. Prieur, and L. Dugard, "Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control," *Automatica*, vol. 49, no. 11, pp. 3180 – 3188, 2013.

[3] E. Schuster, L. Luo, and M. Krstic, "MHD channel flow control in 2D: Mixing enhancement by boundary feedback," *Automatica*, vol. 44, no. 10, pp. 2498 – 2507, 2008.

[4] A. Papachristodoulou and M. Peet, "On the analysis of systems described by classes of partial differential equations," in *Decision and Control, 2006 45th IEEE Conference on*, 2006, pp. 747–752.

[5] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[6] R. Sepulchre, M. Janković, and P. V. Kokotović, *Constructive nonlinear control*, ser. Communications and Control Engineering. Springer, 1997.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA: SIAM, Jun. 1994, vol. 15.

[8] C. Ebenbauer and F. Allgöwer, "Analysis and design of polynomial control systems using dissipation inequalities and sum of squares," *Computers & Chemical Engineering*, vol. 30, no. 10-12, pp. 1590 – 1602, 2006.

[9] C. Prieur and F. Mazenc, "ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws," *Mathematics of Signals, Controls and Systems*, vol. 24, pp. 111–134, 2012.

[10] F. Mazenc and C. Prieur, "Strict Lyapunov functionals for nonlinear parabolic partial differential equations," in *18th IFAC World Congress*, Milan, Italy, 2011.

[11] ——, "Strict Lyapunov functions for semilinear parabolic partial differential equations," *Mathematical Control and Related Fields*, vol. 1, no. 2, pp. 231–250, 2011.

[12] F. Bribiesca Argomedo, C. Prieur, E. Witrant, and S. Bremond, "A strict control Lyapunov function for a diffusion equation with time-varying distributed coefficients," *Automatic Control, IEEE Transactions on*, vol. 58, no. 2, pp. 290–303, 2013.

[13] S. Dashkovskiy and F. Mironchenko, "Input-to-state stability of infinite-dimensional control systems," *Mathematics of Signals, Controls and Systems*, vol. 25, pp. 1–35, 2013.

[14] G. Valmorbida, M. Ahmadi, and A. Papachristodoulou, "Semi-definite programming and functional inequalities for distributed parameter systems," in *53rd Conference on Decision and Control*, Los Angeles, CA, 2014.

[15] E. Sontag, "Smooth stabilization implies coprime factorization," *Automatic Control, IEEE Transactions on*, vol. 34, no. 4, pp. 435–443, Apr 1989.

[16] A. Sharma, J. Morrison, B. McKeon, D. Limebeer, W. Koberg, and S. Sherwin, "Relaminarisation of $Re_\tau = 100$ channel flow with globally stabilising linear feedback control," *Physics of Fluids*, vol. 23, no. 125105, 2011.

[17] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351 – 359, 1995.

[18] W. M. Haddad and V. S. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-based Approach*. Princeton, New Jersey,: Princeton University Press, 2008.

[19] Z. Luo, B. Guo, and O. Murgol, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, ser. Communications and Control Engineering. London, UK: Springer-Verlag, 1999.

[20] P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, 2000.

[21] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *5th European Control Conference*, Karlsruhe, Germany, 1999.

[22] J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*. Imperial College Press, London, 2009.

[23] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, 2nd ed., ser. Applied Mathematical Sciences. Berlin: Springer-Verlag, 2004, vol. 91.