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Controller Synthesis for Stochastic Systems with Persistent Noise via Semi-definite Programming

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Abstract— **We propose convex controller synthesis algorithms for a class of stochastic differential equations (SDEs) with persistent noise. This includes SDEs in which the noise does not vanish at the equilibria of the system. Our performance criterion is Noise-to-State Stability (NSS) in the moments, which is a generalization of the input-to-state stability (ISS) for SDEs. We formulate synthesis algorithms that, in addition to guaranteeing asymptotic convergence in the case of zero input noise, ensure that an upper bound on the effect of input noise (defined by the Frobenius norm of the noise covariance) is minimized. In the case of linear SDEs, the algorithm is in terms of linear matrix inequalities and, in the case of polynomial data, the method is based on polynomial optimization. The method is illustrated by examples.**

I. INTRODUCTION

For a large class of systems, the complexity and/or the uncertainty in dynamics is modeled by stochastic differential equations (SDEs). Examples are the dynamics of biochemical reactions [1] and the fluctuations of the stock prices [2]. Similar to deterministic systems, Lyapunov methods can be used to study different stability and convergence properties (e.g. almost sure stability and stability in probability) of SDEs [3], [4].

Similarly important is the controller synthesis problem of stochastic systems. Among the early attempts based on solutions of algebraic Riccati equations, one can cite [5], [6]. Inspired by the advances in methods using control Lyapunov functions [7] and constructive Lyapunov stabilization [8] for nonlinear deterministic systems, researchers turned to controller synthesis algorithms for SDEs using stochastic control Lyapunov functions [9].

In the past decade, the developments in polynomial optimization and sum-of-squares (SOS) programming [10] have provided algorithmic methods to construct Lyapunov functions [11] for deterministic systems with polynomial vector fields. In [12], [13], convex controller synthesis algorithms were formulated using polynomial Lyapunov functions. Synthesis methods with rational Lyapunov functions and higher degree Lyapunov functions were proposed in [14] and [15], respectively. A method based on quadratic stabilization was proposed in [16] to design saturating and un-saturating nonlinear feedback control laws for polynomial systems. Controller synthesis with performance bounds was also considered in [12] and [17] and controller synthesis for discontinuous dynamical systems was studied in [18]. In the context of stochastic systems, [19] and [20] are the only attempts using SOS programming to synthesize controllers, where the authors use a logarithmic transformation to obtain a linear version of the Hamilton-Jacobi-Bellman (HJB) equation.

However, these methods do not apply to SDEs subject to persistent noise, where the non-decaying noise exists even at the equilibria of the system. For this class of systems, persistent noise precludes stochastic stability. In addition, notions such as input-to-state stability (ISS) [21], which establish bounds on the state in terms of the supremum norm of a persistent input, can not be applied. This stems from the fact that, for a stochastic input, the Itô integral has infinite variations, while the integral for an input in ISS has finite variation. To overcome this technical difficulty, we require noise-to-state stability (NSS) [22] and/or noise-to-state stability in the pth-moment (pth-NSS) [23], which are generalizations of ISS to stochastic systems. Yet, no algorithmic method was proposed in [22] and [23] to verify (or to design controllers ensuring) NSS or pth-NSS for a given system.

In this paper, we present a set of linear matrix inequalities (LMIs) and SOS conditions that, if satisfied, ensure that an underlying system is NSS or pth-NSS. Equipped with these conditions, we formulate convex controller synthesis algorithms that not only ensure stochastic asymptotic stability for the system without input noise, but also ensure that the moments of the system remain bounded for sufficiently large time. We propose optimization problems that minimize bounds on the effect of input noise covariance. For linear SDEs, our method provides sufficient conditions in terms of LMIs, and NSS is ensured in the mean-square sense [24] or the 2nd-moment. For polynomial SDEs, the method is based on SOS programming and guarantees the boundedness of higher moments. We illustrate the proposed method by two examples.

The rest of the paper is organized as follows. The next section considers some background material on SDEs and NSS. In Section III, we present LMI conditions to synthesize 2nd-NSS controllers. In Section IV, we propose a method based on polynomial optimization to design NSS control laws. The proposed methodology is illustrated by two examples in Section V. Finally, Section VI concludes the paper and gives directions for future research.

Notation:

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ and the set of nonnegative reals by $\mathbb{R}_{\geq 0}$. The n-dimensional set of positive integers is denoted by $\mathbb{N}^n$, and the n-dimensional space of non-negative integers is denoted by $\mathbb{N}^n_{\geq 0}$. The

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notation M' denotes the transpose of matrix M, a symmetric $n \times n$ matrix is denoted $\mathbb{S}^n$, and $Tr\{M\}$ is the trace of the square matrix M. A domain Ω is an open subset of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary $\partial\Omega$. The space of k-times continuous differentiable functions defined on Ω is denoted by $\mathcal{C}^k(\Omega)$ and the space of $\mathcal{C}^k(\Omega)$ functions mapping to a set Γ is denoted $\mathcal{C}^k(\Omega \to \Gamma)$. The space of q-th power integrable functions u defined over Ω is denoted $\mathcal{L}^q(\Omega)$ endowed with the norm $\|u\|_{\mathcal{L}^q(\Omega)} = \left(\int_\Omega |u|^q \, \mathrm{d}x\right)^{\frac{1}{q}}$, for $1 \le q < \infty$, and $\|u\|_{\mathcal{L}^\infty(\Omega)} = \sup_{x\in\Omega} |u|$. For a function $f \in \mathcal{C}^1(\Omega)$ and $g \in \mathcal{C}^2(\Omega)$, ∇f denotes the gradient vector and $\nabla^2 g$ denotes the Hessian matrix. For a random variable X, $\mathrm{E}[X]$ denotes its expected value. For a symmetric matrix function $S(x)$, we define $\lambda_m(S) = \inf_{x\in\Omega} |\lambda_{min}(S(x))|$, where $\lambda_{min} : \mathbb{S}^n \to \mathbb{R}$ is the minimum eigenvalue function. Similarly, $\lambda_M(S) = \sup_{x\in\Omega} |\lambda_{max}(S(x))|$, where $\lambda_{max} : \mathbb{S}^n \to \mathbb{R}$ is the maximum eigenvalue function. For a real matrix A, $\|A\|_F = \sqrt{Tr(AA')}$ denotes its Frobenius norm. For two functions f and g, $f \circ g$ is the composition of f and g. For a set S, $Cd(S)$ denotes the cardinality of S or the number of elements in S. For an $m \times n$ matrix A, the notation A_j denotes the jth row of A. A continuous, strictly increasing, function $k : [0, a) \to \mathbb{R}_{\ge 0}$, satisfying $k(0) = 0$, belongs to class $\mathcal{K}$. If $a = \infty$ and $\lim_{x\to\infty} k(x) = \infty$, k belongs to class $\mathcal{K}_\infty$. A function $k(s,t)$ belongs to class $\mathcal{KL}$, if $k(s,\cdot) \in \mathcal{K}$ and $\lim_{t\to\infty} k(\cdot, t) = 0$.

II. Preliminaries and Definitions

Let $(\Gamma, \mathcal{J}, \{\mathcal{J}_t\}_{t\ge0}, \mathrm{P})$ be a complete and right-continuous filtered probability space, where Γ is a sample space, $\{\mathcal{J}_t\}_{t\ge0}$ with $\mathcal{J}_t \subseteq \mathcal{J}$ for each t is a filteration of the σ-algebra $\mathcal{J}$, and P is the probability measure function. Consider the following SDE

$$\mathrm{d}x(t) = f(x(t), t)\, \mathrm{d}t + G(x(t), t)\Sigma(t)\, \mathrm{d}W(t) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state. The initial condition is given by $x(0) = x_0$ with probability 1 for some $x_0 \in \mathbb{R}^n$. The functions $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$, $G : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n\times q}$ and $\Sigma : [0, \infty) \to \mathbb{R}^{q\times m}$ are measurable. The functions f and G correspond to the system dynamics, whereas Σ determines a linear transformation of the m-dimensional Brownian motion $\{W(t)\}_{t\ge0}$, so that at time $t \ge 0$ the stochastic input to the system is the process $\{\Sigma(t)W(t)\}_{t\ge0}$, with covariance $\int_0^t \Sigma(\tau)\Sigma'(\tau)\, \mathrm{d}\tau$.

An $\mathbb{R}^n$-valued random process $\{x(t)\}_{t\ge0}$ is a solution of (1) with initial condition x_0, if

- it is continuous with probability 1, $\{\mathcal{J}_t\}$-adapted, and satisfies $x(0) = x_0$ with probability 1,
- the process $\{f(x(t), t)\}_{t\ge0}$ and $\{G(x(t), t)\}_{t\ge0}$ belong to $\mathcal{L}^1([0, \infty) \to \mathbb{R}^n)$ and $\mathcal{L}^2([0, \infty) \to \mathbb{R}^{n\times m})$, respectively, and
- Equation (1) holds for every $t \ge 0$ with probability 1.

The following assumptions ensure the existence and uniqueness of the solutions to (1).

Assumption 1 ([3], Theorem 3.6, p. 14): Let Σ be locally essentially bounded. In addition, for all $T > 0$ and $n \ge 1$, there exists $K(T, n) > 0$ such that, for almost every $t \in [0, T]$ and all $x, y \in \mathbb{R}^n$ with $\max\{\|x\|_2, \|y\|_2\} \le n$,

$$\max\left\{\|f(x,t) - f(y,t)\|_2^2, \|G(x,t) - G(y,t)\|_F^2\right\} \le K(T, n)\|x - y\|_2^2. \qquad (2)$$

Finally, for all $T > 0$, there exists $C(T) > 0$ such that for almost every $t \in [0, T]$ and all $x, y \in \mathbb{R}^n$, $x'f(x,t) + \frac{1}{2}\|G(x,t)\|_F^2 \le C(T)(1 + \|x\|_2^2)$.

In this paper, we study noise-to-state stability in the p-th moment (pth-NSS), which is defined next.

Definition 1 (pth-NSS): System (1) is pth-NSS, if there exists $\beta \in \mathcal{KL}$ and $\theta \in \mathcal{K}$ such that

$$\mathrm{E}\left[|x(t)|^p\right] \le \beta\left(|x_0|^p, t\right) + \theta\left(ess \sup_{\tau\in[0,t)} \|\Sigma(\tau)\|_F\right), \quad \forall t \ge 0, \ \forall x_0 \in \mathbb{R}^n. \qquad (3)$$

Note that the Frobenius norm of $\Sigma(t)$ is a measure of the size of the noise as it is related to the infinitesimal covariance $\Sigma(t)\Sigma'(t)$.

In order to provide computational tools for the analysis of system (1), we use NSS Lyapunov functions as defined next.

Definition 2 (NSS Lyapunov Function): A function $V \in \mathcal{C}^2(\mathbb{R}^n \to \mathbb{R}_{\ge0})$ is an NSS Lyapunov function, if there exists $W \in \mathcal{C}(\mathbb{R}^n \to \mathbb{R}_{\ge0})$, $\sigma \in \mathcal{K}$ and concave $\eta \in \mathcal{K}_\infty$ such that

$$V(x) \le \eta\left(W(x)\right), \qquad \forall x \in \mathbb{R}^n, \qquad (4)$$

and the following dissipation inequality is satisfied

$$\mathcal{L}[V](x) \le -W(x) + \sigma\left(\|\Sigma(t)\|_F\right), \qquad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_{\ge0}, \qquad (5)$$

where

$$\mathcal{L}[V](x) = (\nabla V)' f(x,t) + \frac{1}{2} Tr\left(G'(x,t)\Sigma'(t)\nabla^2 V \Sigma(t) G(x,t)\right). \qquad (6)$$

is the Itô derivative of V.

The next result generalizes Theorem 4.1 in [22] and will be used in the sequel.

Theorem 2 ([23], Theorem 3.6): Under Assumption 1 and further assuming that $\Sigma(t)$ being continuous, let V be an NSS Lyapunov function for (1). Then, it holds that

$$\mathrm{E}\left[V(x(t))\right] \le \tilde{\beta}\left(V(x_0), t\right) + \eta\left(2\sigma\big(\sup_{\tau\in[0,t)} \|\Sigma(\tau)\|_F\big)\right), \quad \forall t \ge 0, \qquad (7)$$

where $\tilde{\beta} \in \mathcal{KL}$ is the solution to the Cauchy problem

$$\dot{y}(t) = -\frac{1}{2}\eta^{-1}\left(y(t)\right), \qquad y(0) = y_0. \qquad (8)$$

Note that the NSS property (7) ensures that (since $\tilde{\beta} \in \mathcal{KL}$) the system is asymptotically stable in the absence of input noise. Moreover, as $t \to \infty$, inequality (7) reduces to

$$\lim_{t\to\infty} \mathrm{E}\left[V(x(t))\right] \le \tilde{\eta}\left(\sup_{\tau\in[0,\infty)} \|\Sigma(\tau)\|_F\right), \ \forall t \ge 0, \qquad (9)$$

where $\tilde{\eta} = \eta \circ 2 \circ \sigma$. Thus, if the supremum of $\|\Sigma(\tau)\|_F$ is bounded, then the moments of the system remain bounded, as well. This parallels the notion of ISS for deterministic systems.

However, to study pth-NSS, we require the existence of strong NSS Lyapunov functions.

Definition 3 (pth-NSS Lyapunov Functions): A function $V \in \mathcal{C}^2(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ is a strong NSS Lyapunov function in probability for system (1), if V is an NSS Lyapunov function and, in addition, there exists $p > 0$ and $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\alpha_1 (|x|^p) \leq V(x) \leq \alpha_2 (|x|^p), \quad \forall x \in \mathbb{R}^n. \tag{10}$$

If α_1 is convex, then V is a pth-NSS Lyapunov function with respect to Ω.

The next Theorem establishes that the existence of an NSS Lyapunov function implies NSS.

Theorem 3 ([23], Corollary 3.9): Under Assumption 1, and further assuming that Σ is continuous, if $V : \mathcal{C}^2(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ is a pth-NSS Lyapunov function for system (1), then the system is pth-NSS with β and θ as given below

$$\beta(r,s) = \alpha_1^{-1}\left(2\bar{\beta}(\alpha_2(r^p), s\right), \; \theta(r) = \alpha_1^{-1}\left(2\eta(2\sigma(r))\right).$$

In the following sections, we present conditions based on semi-definite programming to synthesize controllers with a guaranteed bound.

III. 2nd-Noise-to-State Stabilizing Controller Synthesis for Linear SDEs

Consider the following linear SDE

$$\mathrm{d}x = (Ax + Bu)\ \mathrm{d}t + G\Sigma(t)\ \mathrm{d}W(t), \tag{11}$$

where $x(0) = x_0 \in \mathbb{R}^n$ with probability 1. The matrices $A \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n_u}$, $G \in \mathbb{R}^{n\times q}$ and $\Sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q\times m}$. The following theorem presents LMI conditions which ensure 2nd-NSS of solutions to system (11).

Theorem 4: Consider system (11) with $u \equiv 0$. Assume $\Sigma(t)$ is continuous. If there exist symmetric matrices $P \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times m}$ such that

$$P > 0, \quad F > 0, \quad H > 0, \tag{12}$$

$$A'P + PA + H \leq 0, \tag{13}$$

and

$$G'PG - F \leq 0, \tag{14}$$

then the solutions to (11) are 2nd-NSS and satisfy

$$\mathrm{E}\left[|x(t)|^2\right] \leq \beta\left(|x_0|^2, t\right) + \theta\left(\sup_{\tau\in[0,t)} \|\Sigma(\tau)\|_F\right), \quad \forall t \geq 0, \; \forall x_0 \in \mathbb{R}^n. \tag{15}$$

with

$$\beta(r,s) = \frac{2\lambda_M(P)\lambda_m(H)|r|^2}{\lambda_m(P)} e^{\frac{-s}{2\lambda_M(P)}}, \quad \theta(r) = \frac{\lambda_M(F)}{\lambda_m(P)}|r|^2 \tag{16}$$

Proof: Let $V(x) = x'Px$ be the candidate 2nd-NSS Lyapunov function. From $P > 0$ in (12), it follows that

$$\lambda_m(P)|x|^2 \leq V(x) \leq \lambda_M(P)|x|^2.$$

Thus, inequality (10) is satisfied with $\alpha_1(\cdot) = \lambda_m(P) \times (\cdot)$ being convex and $\alpha_2(\cdot) = \lambda_M(P) \times (\cdot)$. Calculating the Itô derivative of V yields

$$\begin{aligned}\mathcal{L}[V](x) &= x'(A'P + PA)x + \frac{1}{2}Tr\Big(\Sigma'G'(P+P)G\Sigma\Big) \\ &= x'(A'P + PA)x + Tr\Big(\Sigma'G'PG\Sigma\Big)\end{aligned} \tag{17}$$

From inequality (13) and (14), we can infer that

$$\begin{aligned}\mathcal{L}[V](x) &= x'(A'P + PA)x + Tr\Big(\Sigma'G'PG\Sigma\Big) \\ &\leq -x'Hx + Tr\left(\Sigma(t)F^2\Sigma(t)'\right).\end{aligned}$$

Then, we have

$$\mathcal{L}[V](x) \leq -\lambda_m(H)|x|^2 + \lambda_M(F)\|\Sigma(t)\|_F^2.$$

Thus, (5) holds with $W(r) = \lambda_m(H)r^2$ and $\sigma(r) = \lambda_M^2(F)r^2$. Finally, from Theorem 3, it follows that the system is 2nd-NSS and satisfies (15) with functions β and θ as given in (16). ∎

At this point, we study the existence of a 2nd-NSS controller in feedback form $u = Kx$.

Corollary 1: Consider system (11). Assume $\Sigma(t)$ is continuous. Given $0 \leq \alpha \leq 1$ and $\epsilon > 0$, if there exist symmetric matrices $L \in \mathbb{R}^{n_u\times n_u}$, $P \in \mathbb{R}^{n\times n}$ $H \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times m}$ that solve the following minimization problem

$$\text{minimize } \lambda_1 - \alpha\lambda_2$$
$$\text{subject to}$$
$$\lambda_1 > 0, \; \lambda_2 > 0, \tag{18}$$
$$F > 0, \; F - \lambda_1 I \leq 0, \; \lambda_2 I - P \leq 0, \; \epsilon I - H \leq 0 \tag{19}$$
$$\begin{bmatrix} A'P + PA + H & PB \\ B'P & \frac{1}{2}L \end{bmatrix} \leq 0, \tag{20}$$

and (14), then the control law $u = Kx = -L^{-1}B'Px$ renders the system 2nd-NSS as in (15) with

$$\beta(r,t) = \frac{2\epsilon\lambda_M(P)|r|^2}{\lambda_m(P)} e^{\frac{-t}{2\lambda_M(P)}}, \quad \theta(r) = \frac{\lambda_1}{\lambda_2}|r|^2.$$

Proof: Substituting $u = -L^{-1}B'Px$ in (11) yields the closed loop dynamics

$$\mathrm{d}x = \left(A - BL^{-1}B'P\right) x\ \mathrm{d}t + G\Sigma(t)\ \mathrm{d}B(\omega, t).$$

Defining $A_c = A - BL^{-1}B'P$ and replacing A with A_c in (13), we obtain

$$(A - BL^{-1}B')'P + P(A - BL^{-1}B') + H \leq 0.$$

That is,

$$A'P + PA + H - 2PBL^{-1}B'P \leq 0.$$

Applying Schur's complement to (20), we see that inequality (13) is satisfied for the closed loop system. Moreover, inequality (19) assures that (12) is also satisfied. Then, with (14) satisfied as well, the system is 2nd-NSS according to Theorem 4 and satisfies (15) with $\theta(r)$ and $\beta(r,t)$ as in (16).

To minimize $\theta(r)$, we need to minimize $\frac{\lambda_M(F)}{\lambda_m(P)}$. From (19) and the fact that $\lambda_1 > 0$ and $F > 0$, we have

$$\lambda_M(F) \le \lambda_1,\ \lambda_m(P) \ge \lambda_2.$$

Then, $\frac{\lambda_M(F)}{\lambda_m(P)} \le \frac{\lambda_1}{\lambda_2}$. Thus, minimizing λ_1 while maximizing λ_2, minimizes $\frac{\lambda_M(F)}{\lambda_m(P)}$. This completes the proof. ∎

IV. Noise-to-State Stabilizing Controller Synthesis for Polynomial SDEs

Consider the following polynomial SDE

$$\mathrm{d}x = (A(x)\zeta(x) + B(x)u)\ \mathrm{d}t + G(x)\Sigma(t)\ \mathrm{d}W(t) \quad (21)$$

with $x(0) = x_0 \in \mathbb{R}^n$ with probability 1, $\zeta : \mathbb{R}^n \to \mathbb{R}^q$ is a vector of monomials in x, $A : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $B : \mathbb{R}^n \to \mathbb{R}^{n\times n_u}$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times s}$, and $\Sigma : \mathbb{R}_{\ge 0} \to \mathbb{R}^{s\times m}$.

Remark 1: Note that for a given polynomial system the representation (21) is not unique, i.e., $A(x)$ and $B(x)$ can be different monomial factorization $\zeta(x)$. Lemma 1 in [15] describes a method to find possible $A(x)$ and $B(x)$ matrices for a given $\zeta(x)$.

The next theorem gives conditions under which system (21) without control $u \equiv 0$ is NSS.

Theorem 5: Consider system (21) with $u \equiv 0$. Let Assumption 1 hold. If there exist polynomial matrices $P : \mathbb{R}^n \to \mathbb{S}^q$, $H : \mathbb{R}^n \to \mathbb{R}^{q\times q}$ and $F : \mathbb{R}^n \to \mathbb{R}^{s\times s}$ such that

$$P(x) > 0, \quad F(x) \ge 0, \quad H(x) > 0, \quad (22)$$

$$A'(x)\left(\nabla\zeta(x)\right)' P(x) + P(x)\left(\nabla\zeta(x)\right) A(x) + \sum_{j=1}^{n} \frac{\partial P(x)}{\partial x_j}(A(x)\zeta(x))_j + H(x) \le 0, \quad (23)$$

and

$$G'(x)\Psi(x)G(x) - F(x) \le 0, \quad (24)$$

where

$$\Psi(x) = 2\Xi_1(x) + 2\left(\nabla\zeta(x)\right)' P(x)\left(\nabla\zeta(x)\right) + \Xi_2(x),$$

$$\Xi_1(x) = \left(\frac{\partial\left(\nabla\zeta(x)\right)'}{\partial x_1} P(x)\zeta(x), \cdots, \zeta'(x)P(x)\frac{\partial\left(\nabla\zeta(x)\right)}{\partial x_1}\right) \in \mathbb{R}^{n\times n},$$

$$\Xi_2(x) = \left(\nabla\zeta(x)\right)' Q(x) + S(x),$$

$$Q(x) = \left(\frac{\partial P(x)}{\partial x_1}\zeta(x), \cdots, \frac{\partial P(x)}{\partial x_n}\zeta(x)\right) \in \mathbb{R}^{q\times n},$$

$$S(x) = \left(\frac{\partial Q'(x)}{\partial x_1}\zeta(x), \cdots, \frac{\partial Q'(x)}{\partial x_n}\zeta(x)\right) \in \mathbb{R}^{n\times n},$$

$$\frac{\partial Q'(x)}{\partial x_i} = \left(\frac{\partial^2 P(x)}{\partial x_1 \partial x_i} + \frac{\partial P(x)}{\partial x_1}\frac{\partial\zeta(x)}{\partial x_1}, \cdots, \frac{\partial^2 P(x)}{\partial x_1 \partial x_n} + \frac{\partial P(x)}{\partial x_n}\frac{\partial\zeta(x)}{\partial x_n}\right),$$

then the system is NSS and satisfies

$$\mathrm{E}\left[|\zeta(x)|^2\right] \le \frac{\lambda_M(P(x))\lambda_m(H(x))|\zeta(x_0)|^2}{\lambda_m(P(x))} e^{\frac{-t}{2\lambda_M(P(x))}} + \frac{\lambda_M\left(F(x)\right)}{\lambda_m(P(x))}\left(\sup_{\tau\in[0,t)} \|\Sigma(\tau)\|_F\right)^2, \quad \forall t \ge 0. \quad (25)$$

Proof: This is a result of applying Theorem 2 with NSS-Lyapunov function candidate $V(x) = \zeta'(x)P\zeta(x)$. The detailed proof is omitted here. ∎

Remark 2: As stated in Remark 1, we can find a corresponding $A(x)$ for fixed $\zeta(x)$. This allows us to find bounds on higher moments of a given system. For instance, for a two state stochastic system, if $\zeta(x) = [x_1^2\ x_2^2]'$, then inequality (25) provides bounds on the 4th-moment as follows

$$\mathrm{E}\left[|x|^4\right] \le \frac{\lambda_M(P(x))\lambda_m(H(x))|x_0|^4}{\lambda_m(P(x))} e^{\frac{-t}{2\lambda_M(P(x))}} + \frac{\lambda_M\left(F(x)\right)}{\lambda_m(P(x))}\left(\sup_{\tau\in[0,t)} \|\Sigma(\tau)\|_F\right)^2, \quad \forall t \ge 0. \quad (26)$$

The next result hinges on Theorem 5 which provides a controller synthesis algorithm ensuring NSS. Let $\bar{x} = (x_{j_1}, \ldots, x_{j_m})'$, where $(j_1, \ldots, j_m) \in J$ is the set of indices corresponding to the zero row of $B(x)$.

Corollary 2: Consider system (21). Let Assumption 1 hold. Given $0 < \alpha \le 1$ and $\epsilon > 0$, if there exist matrices $L : \mathbb{R}^n \to \mathbb{R}^{n_u\times n_u}$, $H : \mathbb{R}^n \to \mathbb{R}^{q\times q}$ and $P : \mathbb{R}^{Cd(J)} \to \mathbb{R}^{q\times q}$, and $F : \mathbb{R}^n \to \mathbb{R}^{s\times s}$ such that the minimization problem is feasible

$$\text{minimize } \lambda_1 - \alpha\lambda_2$$
$$\text{subject to}$$
$$\lambda_1 > 0,\ \lambda_2 > 0, \quad (27)$$
$$F(x) > 0,\ F(x) - \lambda_1 I \le 0, \quad (28)$$
$$\lambda_2 I - P(\bar{x}) \le 0,\ \epsilon I - H(x) \le 0, \quad (29)$$
$$\begin{bmatrix} Z(X) + H(x) & P(\bar{x})\left(\nabla\zeta(x)\right) B(x) \\ B'(x)\left(\nabla\zeta(x)\right)' P(\bar{x}) & \frac{1}{2}L(x) \end{bmatrix} \le 0, \quad (30)$$

where

$$Z(x) = A'(x)\left(\nabla\zeta(x)\right)' P(\bar{x}) + P(\bar{x})\left(\nabla\zeta(x)\right) A(x) + \sum_{j=1}^{n} \frac{\partial P(\bar{x})}{\partial x_j}(A(x)\zeta(x))_j$$

and (24), then there exist a control law

$$u = K(x)\zeta(x) = -L^{-1}(x)B'(x)\left(\nabla\zeta(x)\right)' P(\bar{x})\zeta(x),$$

that renders system (21) NSS satisfying (25) with $\lambda_m(H(x)) = \epsilon$, $\lambda_M(F(x)) = \lambda_1$ and $\lambda_m(P(x)) = \lambda_1$.

Proof: The proof follows the same lines as the proof of Theorem 5 and is removed here due to space limitations. ∎

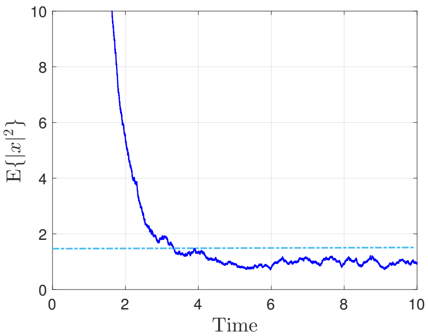

Fig. 1: The evolution of $\mathrm{E}[|x(t)|^2]$. The dashed line illustrates the bound computed $\mathrm{E}[|x|^2] = 1.8945$.

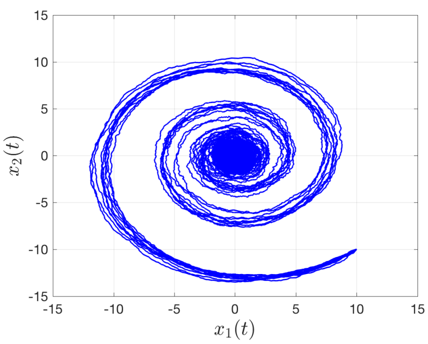

Fig. 2: Phase portrait of ten trajectories of the controlled system in Example I starting at $x(0) = (10, -10)'$

V. EXAMPLES

In this section, we illustrate the proposed controller synthesis algorithms using two examples. The associated LMI problems are solved using YALMIP [25] and the SOS programs are solved using SOSTOOLS [26].

A. *Example I*

Consider the linear SDE (11) with

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix}, \ B = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \ G = I, \ \Sigma = I.$$

Note that the open loop system is unstable and $\|\Sigma\|_F = 1$. For $\alpha = 0.01$ and $\epsilon = 10^{-2}$, using Corollary 1, we obtain a controller $u = Kx$ with

$$K = \begin{bmatrix} -0.4643 & 2.6786 \\ 2.6786 & 2.8929 \end{bmatrix}.$$

The obtained certificates are given as

$$P = \begin{bmatrix} 1.3471 & 0 \\ 0 & 1.3471 \end{bmatrix}, \ H = I, \ F = \begin{bmatrix} 2.6278 & 0 \\ 0 & 2.6278 \end{bmatrix}.$$

Then, $\frac{\lambda_1}{\lambda_2} = 1.8945$. This is consistent with the simulation results as can be seen in Fig. 1 obtained from 50 Monte Carlo simulations. Fig. 2 shows the phase space of ten trajectories of the controlled system.

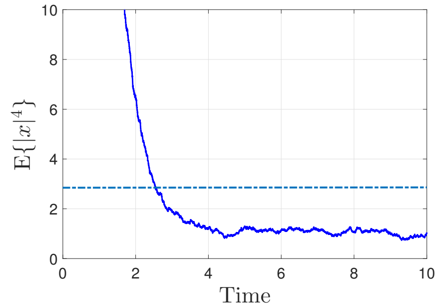

Fig. 3: The evolution of $\mathrm{E}[|x(t)|^4]$. The dahed line illustrates the bound computed $\mathrm{E}[|x|^4] = 2.8329$.

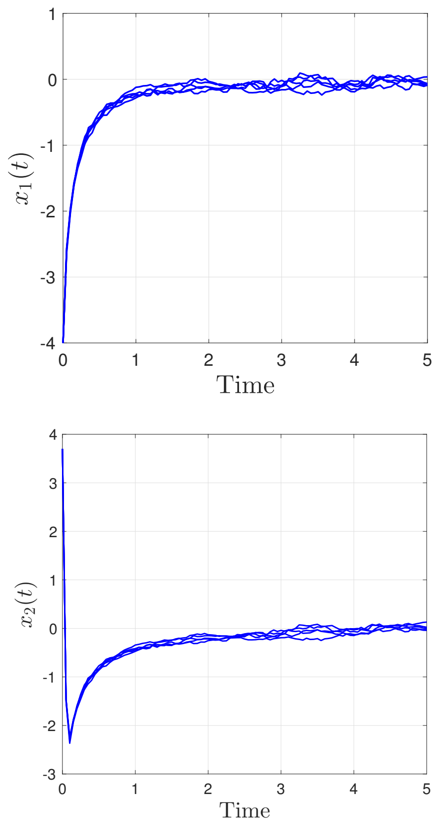

Fig. 4: Five trajectories of the controlled system in Example II starting at $x(0) = (-4, 3)'$

B. *Example II (adapted from [15] with modifications)*

Consider stochastic system (21) with

$$A(x) = \begin{bmatrix} x_1 & 1 + \frac{x_2}{5} \\ x_1^2 - 1 & -x_2 - x_1 x_2 \end{bmatrix}, \ B(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$G(x) = \begin{bmatrix} 0.2 & 0.25 \\ 0.25 & 0.2 \end{bmatrix}, \ \Sigma(t) = \begin{bmatrix} \sin(100\pi t) \\ \cos(100\pi t) \end{bmatrix},$$

where $\zeta(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}'$. Note that $\|\Sigma(t)\|_F = 1$. To control the convergence of the 4th moment, we need to find A and B for $\zeta(x) = \begin{bmatrix} x_1^2 & x_2^2 \end{bmatrix}'$. Following the method in [15], we obtain

$$A(x) = \begin{bmatrix} 2x_1 & 2 + \frac{2x_2}{5} \\ 2x_1^2 - 2 & -2x_2 - 2x_1x_2 \end{bmatrix},\ B(x) = \begin{bmatrix} 0 \\ 2x_2 \end{bmatrix}.$$

Based on Corollary 2, for $\alpha = 1$ and $\epsilon = 1$, we obtain the following controller

$$K(x) = \begin{bmatrix} \frac{-0.649x_2}{2840x_1^2 - 44.7x_1x_2 + 63.8x_1 + 3710x_2 - 13.6x_2 + 4180} \\ \frac{0.818x_2}{2840x_1^2 - 44.7x_1x_2 + 63.8x_1 + 3710x_2 - 13.6x_2 + 4180} \end{bmatrix}',$$

which yields the bound $\frac{\lambda_1}{\lambda_2} = 2.8329$ on the noise covariance. The rest of the certificates are given as

$$P = \begin{bmatrix} 3.8505 & -2.6011 \\ -2.6011 & 3.7793 \end{bmatrix},$$

$$H(x) = \begin{bmatrix} 5.94x_1^2 - 0.434x_1x_2 - 6.25x_1 + 13.2x_2^2 + 0.453x_2 - 2.01 & -3.09x_1^2 - 3.12x_1x_2 + 2.54x_1 - 1.28x_2 - 3.45x_2 - 0.509 \\ -3.09x_1^2 - 3.12x_1x_2 + 2.54x_1 - 1.28x_2 - 3.45x_2 - 0.509 & -0.118x_1^2 + 5.66x_1x_2 - 0.0341x_1 + 0.245x_2 + 6.21x_2 + 5.6 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 4.33x_1^2 - 0.246x_1x_2 + 4.4x_2 + 26.5 & 0.128x_1^2 - 0.276x_1x_2 + 0.149x_2^2 \\ 0.128x_1^2 - 0.276x_1x_2 + 0.149x_2 & 4.39x_1^2 - 0.246x_1x_2 + 4.34x_2^2 + 26.5 \end{bmatrix}.$$

Hence, the controlled system satisfies

$$\mathrm{E}\left[|x(t)|^4\right] \leq 5.2874|x_0|^4 e^{\frac{-t}{12.8324}} + 2.8329, \quad \forall t \geq 0.$$

This is consistent with the simulation results as illustrated in Fig. 3 obtained from 50 Monte Carlo simulations. Fig. 4 depicts five trajectories of the controlled system starting at $x(0) = (-4, 3)'$.

VI. Conclusions

We proposed conditions based on LMIs and SOS programming to verify NSS and/or pth-NSS of linear and polynomial SDEs. These conditions were used to formulate convex controller synthesis algorithms. The method allows us to synthesize controllers that provide bounds on the higher moments of SDEs subject to persistent noise. Future research will consider designing optimal or suboptimal control laws for systems subject to persistent noise and designing controllers that enlarge a guaranteed region of attraction [27] of an SDE using SOS programming (see [28] for the deterministic case).

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