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Brief paper

Dissipation inequalities for the analysis of a class of PDEs☆

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ABSTRACT

In this paper, we develop dissipation inequalities for a class of well-posed systems described by partial differential equations (PDEs). We study passivity, reachability, induced input–output norm boundedness, and input-to-state stability (ISS). We consider both cases of in-domain and boundary inputs and outputs. We study the interconnection of PDE–PDE systems and formulate small gain conditions for stability. For PDEs polynomial in dependent and independent variables, we demonstrate that sum-of-squares (SOS) programming can be used to compute certificates for each property. Therefore, the solution to the proposed dissipation inequalities can be obtained via semi-definite programming. The results are illustrated with examples.

1. Introduction

A powerful tool in the study of robustness and input-to-state/output properties of dynamical systems is dissipation inequalities (Hill & Moylan, 1976; Willems, 1972). A dissipation inequality relates a storage function/functional, which characterizes the internal energy in the system, and a supply rate, which represents a generalized power supply function. Given a supply rate, the solution to the dissipation inequality is a storage function/functional, which according to the supply rate can certify different system properties such as passivity, induced $\mathcal{L}^2$-norm boundedness, reachability, and ISS. One major advantage of dissipation inequalities is that, in the case of systems consisting of an interconnection of subsystems, once some property of the subsystems is known in terms of dissipation inequalities, one can infer properties of the overall system (Van der Schaft, 1996).

For linear systems described by ordinary differential equations (ODEs), quadratic storage functions of states are shown to be both necessary and sufficient solutions to dissipation inequalities with quadratic supply rates (Trentelman & Willems, 1997). For example, the Kalman–Yakubovic–Popov lemma (Kalman, 1963) presents necessary and sufficient conditions to construct quadratic storage functions certifying the passivity dissipation inequality of linear ODE systems. These conditions are given in terms of quadratic expressions, which can be checked computationally via linear matrix inequalities (LMIs) (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994, Chapter 2). For ODEs with polynomial vector fields, an approach to construct polynomial storage functions based on SOS programming has been proposed in Ebenbauer and Allgöwer (2006). For general nonlinear ODEs, however, the solution to dissipation inequalities may require *ad hoc* techniques.

This paper aims at developing dissipation inequalities for systems defined by partial differential equations (PDEs). For PDEs, the solution (state) is a function of both space and time. Moreover, the solution belongs to an infinite-dimensional (function) space, as opposed to a Euclidean space in the case of ODEs. Unlike Euclidean spaces, for function spaces, say Sobolev spaces, different norms are not equivalent (Evans, 2010). Therefore, stability and input-to-state/output properties differ from one norm to another. Despite all these complications, PDEs provide a unique modeling paradigm.

In the context of PDEs, solutions to dissipation inequalities have been proposed recently. For linear time-varying hyperbolic PDEs, the weighted $\mathcal{L}^2$-norm functional was considered as a certificate for ISS in Prieur and Mazenc (2012). ISS storage functionals were suggested in Mazenc and Prieur (2011), for semi-linear parabolic PDEs. In Bribiesca Argomedo, Prieur, Witrant, and Bremond (2013), ISS of a semi-linear diffusion equation was analyzed using the weighted $\mathcal{L}^2$-norm as the storage functional and a

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control approach was formulated for a model of magnetic flux profile in tokamak plasma. In this particular case, the calculation of the storage functional is formulated as the solution of a differential inequality, which is solved using a numerical method. More general ISS definitions were presented in Dashkovskiy and Mironchenko (2013), and a small gain theorem for interconnection of PDEs was formulated.

However, once a dissipation inequality is formulated for an input-to-state/output property characterized by a supply rate, solving the dissipation inequality is difficult in general. In this paper, we build on the results in Papachristodoulou and Peet (2006), where the use of SOS programs and computationally efficient methods for stability analysis of a class of linear parabolic PDEs were reported. In Valmorbida, Ahmadi, and Papachristodoulou (in press), we proposed a methodology to solve integral inequalities involving functions specified by a set of boundary conditions using SOS optimization cast as semi-definite programs (SDPs). This includes inequalities encountered in stability and input–output analysis of PDEs using Lyapunov functionals and storage functionals (Ahmadi, Valmorbida, & Papachristodoulou, 2014), respectively.

In this paper, we generalize the results in Ahmadi et al. (2014), wherein dissipation inequalities for different input-state/output properties of PDEs in the space of square integrable functions with in-domain inputs and outputs were studied, and we present a framework for input-state/output analysis of a class of well-posed PDEs. Each input-state/output property, namely passivity, reachability, induced input–output norms and ISS, is defined in the appropriate Sobolev norms. We consider PDEs with in-domain inputs and outputs, and inputs and outputs at the boundaries. Moreover, we study interconnections of PDE–PDEs with interconnection either at the boundary or over the domain. In addition, we use a method based on convex optimization to systematically solve the dissipation inequalities for PDEs described by polynomials of independent and dependent variables. The proposed formulations are illustrated by two examples: the Burgers' equation with nonlinear forcing, and the Kuramoto–Sivashinsky equation.

The paper is organized as follows. The notation used and some preliminary definitions are discussed in the next section. Sections 3 and 4 are concerned with the dissipation inequalities for PDEs with in-domain inputs and outputs and PDEs with boundary inputs and outputs, respectively. Section 5 presents small gain results for the interconnection of PDEs. Section 6 discusses the computational formulation for solving the dissipation inequalities using semi-definite programming. Two examples are given in Section 7 to illustrate the proposed methods. Finally, Section 8 concludes the paper and provides directions for future research.

2. Preliminaries

Notation. The n-dimensional Euclidean space, the space of $n \times n$ symmetric real matrices, the identity matrix, the n-dimensional space of positive integers, and the n-dimensional space of non-negative integers are denoted by $\mathbb{R}^n$, $\mathbb{S}^n$, I, $\mathbb{N}^n$ and $\mathbb{N}_0^n$, respectively. The domain $\Omega \subset \mathbb{R}$ is a connected, open subset of $\mathbb{R}$, and $\overline{\Omega}$ is the closure of set Ω. The boundary $\partial\Omega$ of set Ω is defined as $\overline{\Omega} \setminus \Omega$ with $\setminus$ denoting set subtraction. In this paper, we consider $\Omega = (0, 1)$. Note that any open bounded domain $\Omega' = (a, b) \subset \mathbb{R}$ can be mapped to $\Omega = (0, 1)$ by an appropriate change of variables. The space of k-times continuous differentiable functions defined on Ω is denoted by $\mathcal{C}^k(\Omega)$. Alternatively, $p \in \mathcal{C}^k[x]$ implies p is k-times continuous differentiable in the variable x. If $p \in \mathcal{C}^1$, then $\partial_x p$ is used to denote the derivative of p with respect to variable x, i.e. $\partial_x := \frac{\partial}{\partial x}$. In addition, we adopt Schwartz's multi-index notation. For $u \in (\mathcal{C}^k)^n$, $\alpha \in \mathbb{N}_0^n$, define $D^\alpha u := \left(u_1, \partial_x u_1, \ldots, \partial_x^{\alpha_1} u_1, \ldots, u_n, \partial_x u_n, \ldots, \partial_x^{\alpha_n} u_n\right)$, where $(\mathcal{C}^k)^n$ is the n-dimensional space of C^k functions. The Sobolev space of pth power, up to qth derivative integrable functions u defined over Ω is denoted $\mathcal{W}_\Omega^{q,p}$ endowed with the norm $\|u\|_{\mathcal{W}_\Omega^{q,p}} = \left(\int_\Omega \sum_{i=0}^q \left|\partial_x^i u\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$, for $1 \le p < \infty$ and $q \in \{0, 1, 2, \ldots\}$, and $\|u\|_{\mathcal{W}_\Omega^{q,\infty}} = \max_{i=0,\ldots,q} \left(\sup_{x\in\Omega} \left|\partial_x^i u\right|\right)$, for $p = \infty$, where $|\cdot|$ signifies the absolute value. We denote the case $p = 2$ simply as the Hilbert space $\mathcal{H}_\Omega^q$. For $q = 0$, we use the notation $\mathcal{L}_\Omega^p$ for the Lebesgue space. Also, we use the following notation $\|u\|_{\mathcal{H}^q_{[0,T),\Omega}} = \left(\int_0^T \langle u, u\rangle_{\mathcal{H}_\Omega^q} \,\mathrm{d}t\right)^{\frac{1}{2}}$, where $\langle u, u\rangle_{\mathcal{H}_\Omega^q}$ is the inner product in $\mathcal{H}_\Omega^q$. Whenever the spaces can be inferred from the context, we use $\mathcal{H}^q$ instead of $\mathcal{H}^q_{[0,T),\Omega}$. A continuous, strictly increasing function $k : [0, a) \to \mathbb{R}_{\ge 0}$, satisfying $k(0) = 0$, belongs to class $\mathcal{K}$. If $a = \infty$ and $\lim_{x\to\infty} k(x) = \infty$, k belongs to class $\mathcal{K}_\infty$. We recall that for any class $\mathcal{K}$ function, the inverse exists and belongs to class $\mathcal{K}$. Furthermore, for any positive $a, b > 0$ and $k \in \mathcal{K}$, we have (Sontag, 1989, Inequality (12))

$$k(a + b) \le k(2a) + k(2b). \tag{1}$$

For a symmetric matrix function $S(x)$, we define $\underline{\lambda}(S) = \inf_{x\in\Omega} |\lambda_{\min}(S(x))|$, where $\lambda_{\min} : \mathbb{S}^n \to \mathbb{R}$ is the minimum eigenvalue function. Similarly, $\bar{\lambda}(S) = \sup_{x\in\Omega} |\lambda_{\max}(S(x))|$, where $\lambda_{\max} : \mathbb{S}^n \to \mathbb{R}$ is the maximum eigenvalue function.

Definition 1 (*Stability in $\mathcal{H}_\Omega^q$*). Consider the PDE

$$\partial_t u = F(x, D^\alpha u), \quad x \in \Omega,\ t > 0. \tag{2}$$

Let $\psi(x)$ be an equilibrium of (2), satisfying $F(x, D^\alpha \psi) = 0, x \in \Omega$, and $u(0, x) = u_0(x)$. Then, $\psi(x)$ is

- stable in $\mathcal{H}_\Omega^q$, if for any $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that for all $t \ge 0$
$$\|u_0 - \psi\|_{\mathcal{H}_\Omega^q} < \delta \Rightarrow \|u - \psi\|_{\mathcal{H}_\Omega^q} < \varepsilon,$$
- asymptotically stable in $\mathcal{H}_\Omega^q$, if it is stable and $\exists \delta > 0$ such that
$$\|u_0 - \psi\|_{\mathcal{H}_\Omega^q} < \delta \Rightarrow \lim_{t\to\infty} \|u - \psi\|_{\mathcal{H}_\Omega^q} = 0,$$
- exponentially stable in $\mathcal{H}_\Omega^q$, if there exists a scalar $\lambda > 0$, such that for all $t \ge 0$
$$\|u - \psi\|^2_{\mathcal{H}_\Omega^q} \le \|u_0 - \psi\|^2_{\mathcal{H}_\Omega^q} e^{-\lambda t}.$$

In the sequel, we consider stability to the null solution, i.e. $\psi(x) = 0$, $\forall x \in \Omega$ in Definition 1.

3. PDEs with in-domain inputs and in-domain outputs

In this section, we consider the class of PDE systems described by

$$\begin{cases} \partial_t u(t, x) = F\left(x, D^{\alpha_u} u(t, x), D^{\alpha_d} d(t, x)\right), \\ y(t, x) = H(x, D^\delta u), \quad (t, x) \in \mathbb{R}_{\ge 0} \times \Omega, \\ Q \begin{bmatrix} D^{\alpha_u - 1} u(t, 1) \\ D^{\alpha_u - 1} u(t, 0) \end{bmatrix} = 0, \qquad Q \begin{bmatrix} D^{\alpha_d - 1} d(t, 1) \\ D^{\alpha_d - 1} d(t, 0) \end{bmatrix} = 0, \end{cases} \tag{3}$$

and initial conditions $u(0, x) = u_0(x)$, that admit well-posed solutions. The dependent variables $u = (u_1, u_2, \ldots, u_{n_u})'$, $d = (d_1, d_2, \ldots, d_{n_d})'$, and $y = (y_1, y_2, \ldots, y_{n_y})'$ (defined over both space and time) represent states, inputs, and outputs, respectively, and Q is a matrix of appropriate dimension defining the boundary conditions.

In order to study input-state/output properties of system (3), we define each property as follows.

Definition 2. A. Passivity: System (3) satisfies the following inequality

$$\langle d, y\rangle_{\mathcal{L}^2_{[0,\infty),\Omega}} \geq 0, \quad (4)$$

subject to $u_0(x) \equiv 0,\ \forall x \in \Omega$.

B. $\mathcal{H}^p$-to-$\mathcal{H}^q_\Omega$ Reachability: For $d \in (\mathcal{H}^p)^{n_d}$ with $\alpha_d \geq p$, the solutions of (3) satisfy

$$\|u(T,x)\|_{\mathcal{H}^q_\Omega} \leq \beta\left(\|d(t,x)\|_{\mathcal{H}^p_{[0,T),\Omega}}\right), \quad \forall T > 0 \quad (5)$$

with $\beta \in \mathcal{K}_\infty$ and subject to $u_0(x) \equiv 0,\ \forall x \in \Omega$.

C. Induced $\mathcal{H}^p$-to-$\mathcal{H}^q$-norm Boundedness: For $d \in (\mathcal{H}^p)^{n_d}$ with $\alpha_d \geq p$ and some $\gamma > 0$,

$$\|y\|_{\mathcal{H}^q_{[0,\infty),\Omega}} \leq \gamma \|d\|_{\mathcal{H}^p_{[0,\infty),\Omega}} \quad (6)$$

subject to zero initial conditions $u_0(x) \equiv 0,\ \forall x \in \Omega$.

D. D^p-Input-to-State Stability in $\mathcal{H}^q_\Omega$: For $d \in (\mathcal{W}^{p,\infty}_\Omega)^{n_d}$ with $\alpha_d \geq p$, some scalar $\psi > 0$, functions $\beta, \tilde{\beta}, \chi \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{K}$, it holds that

$$\|u\|_{\mathcal{H}^q_\Omega} \leq \beta\left(e^{-\psi t}\chi\left(\|u_0\|_{\mathcal{H}^q_\Omega}\right)\right) + \tilde{\beta}\left(\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma\left(\left|D^p d(\tau,x)\right|\right)\,\mathrm{d}x\right)\right),\ \forall t > 0. \quad (7)$$

Remark 1. Given $T > 0$ and the information on $\mathcal{H}^p_{[0,T),\Omega}$-norm of input, inequality (5) shows how the state can evolve in the $\mathcal{H}^q_\Omega$ sense at $t = T$. In fact, a minimization over $\beta \in \mathcal{K}_\infty$ results in an upper bound on the reachable set at time $t = T$ in the $\mathcal{H}^q_\Omega$ norm.

Remark 2. In item C in Definition 2, for PDE system (3), we are interested in estimating upper bounds on $\gamma^* > 0$ defined as

$$\gamma^* = \sup_{0<\|d\|_{\mathcal{H}^p}<\infty} \frac{\|y\|_{\mathcal{H}^q}}{\|d\|_{\mathcal{H}^p}}, \quad (8)$$

i.e., the induced $\mathcal{H}^p$-to-$\mathcal{H}^q$-norm.

Remark 3. Note that the D^p-ISS property (7) assures asymptotic convergence to the null solution in $\mathcal{H}^q_\Omega$ when $d \equiv 0$. Moreover, when $d \neq 0$, as $t \to \infty$, the first term on the right-hand side of (7) vanishes yielding

$$\lim_{t\to\infty}\|u(t,x)\|_{\mathcal{H}^q_\Omega} \leq \tilde{\beta}\left(\int_\Omega \|\sigma(|D^p d(t,x)|)\|_{\mathcal{L}^\infty_{[0,\infty)}}\,\mathrm{d}x\right) \leq \tilde{\beta}\left(\int_\Omega \sigma(\|d(t,x)\|_{\mathcal{W}^{p,\infty}_{[0,\infty)}})\,\mathrm{d}x\right), \quad (9)$$

wherein, the fact that $\sigma, \beta \in \mathcal{K}_\infty \subset \mathcal{K}$ is used. Hence, when all the spatial derivatives of the input up to order p are bounded in $\mathcal{L}^\infty_{[0,\infty)}$, the *state* u is bounded in the $\mathcal{H}^q_\Omega$ norm. This is analogous to the ISS property for ODEs (Sontag & Wang, 1995).

Remark 4. The reachability property is often referred to as controllability (Lasiecka, Triggiani, Liu, & Krstić, 2000, Section 9.6.7) and the induced norm boundedness property is often studied in the context of trace regularity (e.g. see the trace regularity results for hyperbolic PDEs Lasiecka et al., 2000, Section 8A and the Schrödinger equation Lasiecka et al., 2000, Section 10.9.3).

In the sequel, we use the concept of *zero-state detectability* for PDEs, which is defined next (for the case of ODEs refer to Haddad & Chellaboina, 2008, p. 362).

Definition 3. A system is zero-state detectable (ZSD) in $\mathcal{H}^q_\Omega$, if $\|y\|_{\mathcal{H}^q_\Omega} = 0$ implies $\|u\|_{\mathcal{H}^q_\Omega} = 0$.

Remark 5. Zero-state detectability imposes constraints on H in (3) ($\|H(x, D^\delta u)\|_{\mathcal{H}^q_\Omega} = 0 \Rightarrow \|u\|_{\mathcal{H}^q_\Omega} = 0$). In the special case of $H(x, D^\delta u) = h(x)u$ and $q = 0$, this is equivalent to $\nexists x \in \Omega$ such that $h(x) = 0$, thereby $y = 0 \Rightarrow u = 0$.

In the next theorem, we formulate the dissipation inequalities associated with properties A–D in Definition 2.

Theorem 6. *Consider the PDE system described by (3). If there exist a positive semidefinite storage functional $S(u)$, scalars $\gamma, \psi > 0$, and functions $\beta_1, \beta_2 \in \mathcal{K}_\infty$, $\alpha, \sigma \in \mathcal{K}$ satisfying $\psi|U| \leq \alpha(|U|)$, such that*

(A)
$$\partial_t S(u) \leq \langle d, y\rangle_{\mathcal{L}^2_\Omega}, \quad (10)$$

(B)
$$\beta_1(\|u\|_{\mathcal{H}^q_\Omega}) \leq S(u), \quad (11)$$
$$\partial_t S(u) \leq \gamma^2\langle d, d\rangle_{\mathcal{H}^p_\Omega}, \quad (12)$$

(C)
$$\|y\|_{\mathcal{H}^q_\Omega} = 0 \Rightarrow \|u\|_{\mathcal{H}^q_\Omega} = 0, \quad (13)$$
$$\partial_t S(u) \leq -\langle y, y\rangle_{\mathcal{H}^q_\Omega} + \gamma^2\langle d, d\rangle_{\mathcal{H}^p_\Omega}, \quad (14)$$

(D)
$$\beta_1(\|u\|_{\mathcal{H}^q_\Omega}) \leq S(u) \leq \beta_2(\|u\|_{\mathcal{H}^q_\Omega}), \quad (15)$$
$$\partial_t S(u) \leq -\alpha(S(u)) + \int_\Omega \sigma(|D^p d|)\,\mathrm{d}x, \quad (16)$$

for all $t > 0$, then, respectively, system (3)

(A) *is passive as in (4),*
(B) *is $\mathcal{H}^p_\Omega$-to-$\mathcal{H}^q_\Omega$ reachable as in (5) with $\beta(\cdot) = \beta_1^{-1}(\gamma(\cdot))$,*
(C) *is asymptotically stable and its induced $\mathcal{H}^p$-to-$\mathcal{H}^q$-norm is bounded by γ as in (6).*
(D) *is D^p-ISS in $\mathcal{H}^q_\Omega$ and satisfies (7) with $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1} \circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1} \circ \frac{2}{\psi}(\cdot)$.*

Proof. Each item is proven in turn:
(A) Integrating both sides of (10) over time from 0 to ∞ yields $\int_0^\infty \partial_t S(u)\,\mathrm{d}t \leq \int_0^\infty \int_\Omega \langle d, y\rangle_{\mathcal{L}^2_\Omega}\,\mathrm{d}t$. That is, $\lim_{t\to\infty} S(u(t,x)) - S(u_0) \leq \int_0^\infty \int_\Omega d'y\,\mathrm{d}x\mathrm{d}t$. By hypothesis, $S(u)$ is positive semidefinite. Hence, for $u(0,x) = 0$, we have $S(u(0,x)) = 0$. Moreover, $\lim_{t\to\infty} S(u(t,x)) \geq 0$. Therefore, we obtain the passivity estimate (4).
(B) Integrating both sides of (12) over time from 0 to T yields $\int_0^T \partial_t S(u)\,\mathrm{d}t \leq \gamma \int_0^T \|d\|^2_{\mathcal{H}^q_\Omega}\,\mathrm{d}t$. That is, $S(u(T,x)) - S(u(0,x)) \leq \gamma\|d(t,x)\|_{\mathcal{H}^p_{[0,T),\Omega}}$. Noting that, with $u(0,x) \equiv 0$, from (11), we have $\beta_1(\|u(T,x)\|_{\mathcal{H}^q_\Omega}) \leq S(u(T,x)) \leq \gamma\|d(t,x)\|_{\mathcal{H}^p_{[0,T),\Omega}}$. Since $\beta_1 \in \mathcal{K}_\infty$, its inverse exists and belongs to $\mathcal{K}_\infty$. Thus, $\|u(T,x)\|_{\mathcal{H}^q_\Omega} \leq \beta_1^{-1}\left(\gamma\|d(t,x)\|_{\mathcal{H}^p_{[0,T),\Omega}}\right)$. Therefore, an estimate of the reachable set at $t = T$ in terms of $\|d\|_{\mathcal{H}^p_{[0,T),\Omega}}$ is attained.
(C) Subject to zero inputs $d \equiv 0$, (14) becomes

$$\partial_t S(u) \leq -\|y\|^2_{\mathcal{H}^q_\Omega}. \quad (17)$$

Inequality (17) implies that the time derivative of the storage functional $S(u)$ is negative semidefinite. Moreover, from Definition 3, condition (13) is equivalent to system (3) being ZSD in $\mathcal{H}^q_\Omega$. Thus, $\partial_t S(u) = 0$ only if $\|u\|_{\mathcal{H}^q_\Omega} = 0$. Hence, from LaSalle's invariance principle (Luo, Guo, & Murgol, 1999, Theorem 3.64, p. 161), it follows that u converges to the null solution $u = 0$ in $\mathcal{H}^q_\Omega$-norm asymptotically.
Furthermore, by integrating both sides of (14) from 0 to ∞, we obtain $\int_0^\infty \partial_t S(u)\,\mathrm{d}t \leq -\int_0^\infty \|y\|^2_{\mathcal{H}^q_\Omega}\,\mathrm{d}t + \gamma^2\int_0^\infty \|d\|^2_{\mathcal{H}^p_\Omega}\,\mathrm{d}t$. That is, $\lim_{t\to\infty} S(u(t,x)) - S(u_0) \leq -\int_0^\infty \|y\|^2_{\mathcal{H}^q_\Omega}\,\mathrm{d}t + \gamma^2\int_0^\infty \|d\|^2_{\mathcal{H}^p_\Omega}\,\mathrm{d}t$. Since $u_0(x) \equiv 0$, $x \in \Omega$, we have $\lim_{t\to\infty} S(u(t,x)) \leq -\int_0^\infty \|y\|^2_{\mathcal{H}^q_\Omega}\,\mathrm{d}t + \gamma^2\int_0^\infty \|d\|^2_{\mathcal{H}^p_\Omega}\,\mathrm{d}t$, and because $S(\cdot)$ is positive semidefinite, we obtain $\int_0^\infty \|y\|^2_{\mathcal{H}^q_\Omega}\,\mathrm{d}t \leq \gamma^2\int_0^\infty \|d\|^2_{\mathcal{H}^p_\Omega}\,\mathrm{d}t$.

(D) By rearranging the terms in (16) and assuming that $\psi|U| \leq \alpha(|U|)$, we have $\partial_t S(u) + \psi S(u) \leq \int_\Omega \sigma(|D^p d|)\,\mathrm{d}x$. Multiplying both sides of the above inequality by the strictly increasing function $e^{\psi t}$, we have $e^{\psi t}(\partial_t S(u) + \psi S(u)) \leq e^{\psi t}\int_\Omega \sigma(|D^p d|)\,\mathrm{d}x$. Then, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\psi t}S\right) \leq e^{\psi t}\int_\Omega \sigma(|D^p d|)\,\mathrm{d}x. \tag{18}$$

Integrating both sides of inequality (18) from 0 to t gives

$$\begin{aligned} &e^{\psi t}S(u(t,x)) - S(u(0,x)) \\ &\leq \int_0^t e^{\psi\tau}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right)\mathrm{d}\tau \\ &\leq \left(\int_0^t e^{\psi\tau}\,\mathrm{d}\tau\right)\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right) \\ &\leq \frac{1}{\psi}(e^{\psi t}-1)\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right) \\ &\leq \frac{e^{\psi t}}{\psi}\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right), \end{aligned} \tag{19}$$

where Hölder's inequality is used in the second inequality above. Dividing both sides of the last inequality above by the positive term $e^{\psi t}$ gives $S(u) \leq e^{-\psi t}S(u_0) + \frac{1}{\psi}\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right)$. Using (15), we infer that

$$\begin{aligned} \beta_1(\|u\|_{\mathcal{H}^q_\Omega}) &\leq e^{-\psi t}\beta_2(\|u_0\|_{\mathcal{H}^q_\Omega}) \\ &\quad + \frac{1}{\psi}\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right). \end{aligned} \tag{20}$$

Since $\beta_1 \in \mathcal{K}_\infty$, its inverse exists and belongs to $\mathcal{K}_\infty$. Hence, taking the inverse of β_1 from both sides of (20) yields $\|u\|_{\mathcal{H}^q_\Omega} \leq \beta_1^{-1}\big(e^{-\psi t}\beta_2(\|u_0\|_{\mathcal{H}^q_\Omega}) + \frac{1}{\psi}\sup_{\tau\in[0,t)}\big(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\big)\big)$, and, applying inequality (1), it follows that

$$\begin{aligned} \|u\|_{\mathcal{H}^q_\Omega} &\leq \beta_1^{-1}\left(2e^{-\psi t}\beta_2(\|u_0\|_{\mathcal{H}^q_\Omega})\right) \\ &\quad + \beta_1^{-1}\left(\frac{2}{\psi}\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\,\mathrm{d}x\right)\right), \end{aligned}$$

and (7) is obtained with $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1}\circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1}\circ\frac{2}{\psi}(\cdot)$.

Remark 7 (*Conservative Systems*)**.** For $p = q = 0$, if we consider the squared $\mathcal{L}^2_\Omega$-norm as the storage functional, i.e. $S(u) = \|u\|^2_{\mathcal{L}^2_\Omega} = \int_\Omega u^2\,\mathrm{d}x$, inequality (14) in Theorem 6 can be re-written as

$$\partial_t\left(\|u\|^2_{\mathcal{L}^2_\Omega}\right) \leq \gamma^2\|d\|^2_{\mathcal{L}^2_\Omega} - \|y\|^2_{\mathcal{L}^2_\Omega}.$$

Integrating both sides of the above inequality over time from 0 to $T > 0$ yields $\|u(T,\cdot)\|^2_{\mathcal{L}^2_\Omega} - \|u_0\|^2_{\mathcal{L}^2_\Omega} \leq \gamma^2\int_0^T \|d\|^2_{\mathcal{L}^2_\Omega}\,\mathrm{d}t - \int_0^T \|y\|^2_{\mathcal{L}^2_\Omega}\,\mathrm{d}t$. Then, in the special case when $\gamma = 1$ and equality holds, we obtain $\|u(T,\cdot)\|^2_{\mathcal{L}^2_\Omega} - \|u_0\|^2_{\mathcal{L}^2_\Omega} = \int_0^T \|d\|^2_{\mathcal{L}^2_\Omega}\,\mathrm{d}t - \int_0^T \|y\|^2_{\mathcal{L}^2_\Omega}\,\mathrm{d}t$, which implies that the PDE is conservative[2] as studied in Weiss, Staffans, and Tucsnak (2001); Weiss and Tucsnak (2003).

We illustrate Theorem 6 using an example.

Example 1. Consider the following PDE system

$$\begin{aligned} &\partial_t u(t,x) = \partial_x^2 u(t,x) - u(t,x)\partial_x u(t,x) + d(t,x), \\ &y(t,x) = u(t,x), \quad x \in (0,1),\ t > 0 \end{aligned} \tag{21}$$

subject to $u(0,t) = u(1,t) = 0$. In the following, we show that for the above system the following storage functional satisfies inequalities (15), and (16) with $p = 0$ and $q = 0$

$$S(u) = \frac{1}{2}\int_0^1 u^2(t,x)\,\mathrm{d}x. \tag{22}$$

In other words, using storage functional (22), we demonstrate that the system is D^0-ISS in $\mathcal{L}^2_\Omega$. Note that the storage functional (22) satisfies $\frac{c}{2}\int_0^1 u^2\,\mathrm{d}x \leq \frac{1}{2}\int_0^1 u^2\,\mathrm{d}x \leq \frac{C}{2}\int_0^1 u^2\,\mathrm{d}x$ for some $0 < c < 1$ and $C > 1$. Thus, inequality (15) is satisfied with $q = 0$. Substituting (22) in (16) with $p = 0$ and noting that $\psi|U| \leq \alpha(|U|)$, we have

$$\begin{aligned} &-\frac{\psi}{2}\int_0^1 u^2\,\mathrm{d}x + \int_0^1 \sigma(|d|)\,\mathrm{d}x \\ &\geq \int_0^1 u\,\overbrace{\left(\partial_x^2 u - u\partial_x u + d\right)}^{\partial_t u}\,\mathrm{d}x. \end{aligned} \tag{23}$$

By integration by parts and using the boundary conditions, we have $\int_0^1 u\partial_x^2 u\,\mathrm{d}x = -\int_0^1 (\partial_x u)^2\,\mathrm{d}x$, and $\int_0^1 u^2\partial_x u\,\mathrm{d}x = 0$. Then, inequality (23) becomes

$$\begin{aligned} -\frac{\psi}{2}\int_0^1 u^2\,\mathrm{d}x + \int_0^1 \sigma(|d|)\,\mathrm{d}x &\geq -\int_0^1 (\partial_x u)^2\,\mathrm{d}x \\ &\quad + \int_0^1 ud\,\mathrm{d}x. \end{aligned} \tag{24}$$

In addition, using Hölder and Young inequalities we have

$$\int_0^1 ud\,\mathrm{d}x \leq \frac{1}{2}\int_0^1 u^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 d^2\,\mathrm{d}x. \tag{25}$$

In the following, we show that the left hand side of (24) is greater than a quantity which is greater than the right hand side of (24). Thus, inequality (24) also holds. Applying inequality (25), we check $-\frac{\psi}{2}\int_0^1 u^2\,\mathrm{d}x + \int_0^1 \sigma(|d|)\,\mathrm{d}x \geq -\int_0^1 (\partial_x u)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 d^2\,\mathrm{d}x$. Moving the terms involving d and u to the left and the right hand side of the above inequality, respectively, gives $-\frac{\psi}{2}\int_0^1 u^2\,\mathrm{d}x + \int_0^1 (\partial_x u)^2\,\mathrm{d}x - \frac{1}{2}\int_0^1 u^2\,\mathrm{d}x \geq -\int_0^1 \sigma(|d|)\,\mathrm{d}x + \frac{1}{2}\int_0^1 d^2\,\mathrm{d}x$. By choosing $\sigma(|d|) = \frac{d^2}{2}$, we obtain $-\left(\frac{\psi+1}{2}\right)\int_0^1 u^2\,\mathrm{d}x + \int_0^1 (\partial_x u)^2\,\mathrm{d}x \geq 0$. From the Poincaré inequality, we infer that if we choose ψ and correspondingly α such that $\frac{\psi+1}{2} \leq \pi^2$, then the above inequality holds. Consequently, we demonstrated using storage functional (22) that system (21) is D^0-ISS in $\mathcal{L}^2_\Omega$.

From the above example, it is evident that finding the storage functional and checking the associated dissipation inequalities is not straightforward.[3] In Section 6, we demonstrate that for PDEs with polynomial data the dissipation inequalities can be solved by convex optimization.

[2] That is, since $T > 0$ is arbitrary, the induced $\mathcal{L}^2$-to-$\mathcal{L}^2$-norm of the system is 1.

[3] It requires applying integration-by-parts and inequalities from functional analysis.

4. PDEs with boundary inputs and boundary outputs

In this section, under well-posedness assumptions, we formulate conditions to study the input–output properties of PDEs with boundary inputs and outputs. Consider the following PDE system

$$\begin{cases} \partial_t u(t,x) = F\left(x, D^{\alpha} u(t,x)\right), \\ y(t) = h\left(D^{\beta} u(t,0)\right), \qquad (t,x) \in \mathbb{R}_{\geq 0} \times \Omega \\ QD^{\alpha-1}u(t,0) = 0, \qquad QD^{\alpha-1}u(t,1) = w(t), \end{cases} \tag{26}$$

and initial conditions $u(0,x) = u_0(x)$, where Q is of appropriate dimension, $y = (y_1, y_2, \dots, y_{n_y})'$, and $w = (w_1, w_2, \dots, w_{n_w})'$. Next, we define input-state/output properties for PDE (26).

Definition 4. A. Passivity (Van der Schaft, 1996): System (26) satisfies the following inequality

$$\langle w, y \rangle_{\mathcal{L}^2_{[0,\infty)}} \geq 0, \tag{27}$$

subject to $u_0(x) \equiv 0,\ \forall x \in \Omega$.

B. $\mathcal{L}^2_{[0,\infty)}$-to-$\mathcal{H}^q_\Omega$ Reachability (Van der Schaft, 1996): For $w \in (\mathcal{L}^2_{[0,\infty)})^{n_w}$, the solutions of (26) satisfy

$$\|u(T,x)\|_{\mathcal{H}^q_\Omega} \leq \beta\left(\|w(t)\|_{\mathcal{L}^2_{[0,T)}}\right), \quad \forall T > 0 \tag{28}$$

with $\beta \in \mathcal{K}_\infty$ and subject to $u_0(x) \equiv 0,\ \forall x \in \Omega$.

C. Induced $\mathcal{L}^2_{[0,\infty)}$-norm Boundedness (Van der Schaft, 1996): For $w \in (\mathcal{L}^2_{[0,\infty)})^{n_w}$ and some $\gamma > 0$,

$$\|y\|_{\mathcal{L}^2_{[0,\infty)}} \leq \gamma \|w\|_{\mathcal{L}^2_{[0,\infty)}} \tag{29}$$

subject to zero initial conditions $u_0(x) \equiv 0,\ \forall x \in \Omega$.

D. Input-to-State Stability in $\mathcal{H}^q_\Omega$: For $w \in (\mathcal{L}^\infty_{[0,\infty)})^{n_w}$, some scalar $\psi > 0$, functions $\beta, \tilde{\beta}, \chi \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{K}$, it holds that

$$\|u\|_{\mathcal{H}^q_\Omega} \leq \beta\left(e^{-\psi t}\chi\left(\|u_0\|_{\mathcal{H}^q_\Omega}\right)\right) + \tilde{\beta}\left(\sup_{\tau\in[0,t)} \sigma\left(|w(\tau)|\right)\right), \quad \forall t > 0. \tag{30}$$

Remark 8. The Input-to-State Stability in $\mathcal{H}^q_\Omega$ property defined above parallels the ISS property for ODE systems as given in Sontag (1989). However, ISS in $\mathcal{H}^q_\Omega$ property for PDEs includes bounds on states u defined in the Sobolev norm of interest $\mathcal{H}^q_\Omega$. This is essential for the ISS analysis of solutions of PDEs since the norms are not equivalent in Sobolev spaces as opposed to Euclidean spaces.

The next result follows from Theorem 6.

Corollary 9. *Consider the PDE system described by (26). If there exist a positive semidefinite storage functional $S(u)$, scalars γ, $\psi > 0$, and functions $\beta_1, \beta_2 \in \mathcal{K}_\infty$, $\alpha, \sigma \in \mathcal{K}$ satisfying $\psi|U| \leq \alpha(|U|)$, such that*

(A) $$\partial_t S(u) \leq w'(t)y(t), \tag{31}$$

(B) $$\beta_1(\|u\|_{\mathcal{H}^q_\Omega}) \leq S(u), \tag{32}$$

$$\partial_t S(u) \leq \gamma^2 w'(t)w(t), \tag{33}$$

(C) $$\partial_t S(u) \leq -y'(t)y(t) + \gamma^2 w'(t)w(t), \tag{34}$$

(D) $$\beta_1(\|u\|_{\mathcal{H}^q_\Omega}) \leq S(u) \leq \beta_2(\|u\|_{\mathcal{H}^q_\Omega}), \tag{35}$$

$$\partial_t S(u) \leq -\alpha(S(u)) + \sigma(|w(t)|), \tag{36}$$

for all $t > 0$, then, respectively, system (26)
(A) satisfies the passivity property (27),
(B) satisfies the $\mathcal{L}^2_{[0,\infty)}$-to-$\mathcal{H}^q_\Omega$ reachability property (28) with $\beta(\cdot) = \beta_1^{-1}(\gamma^2(\cdot)^2)$,

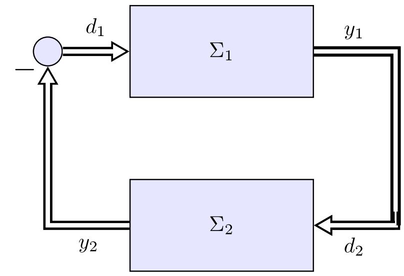

Fig. 1. The interconnection of two PDE systems.

(C) is stable and has its induced $\mathcal{L}^2_{[0,\infty)}$-norm bounded by γ as in (29).
(D) is ISS in $\mathcal{H}^q_\Omega$ and satisfies (30) with $\chi = \beta_2$, $\beta(\cdot) = \beta_1^{-1} \circ 2(\cdot)$ and $\tilde{\beta}(\cdot) = \beta_1^{-1} \circ \frac{2}{\psi}(\cdot)$.

Proof. The proof of Items *A*, *B*, and *D* follows the same lines as the proof of Theorem 6. For Item *C*, LaSalle's invariance principle cannot be used to conclude asymptotic stability as was the case in Theorem 6, since with $d \equiv 0$ inequality (34) is converted to $\partial_t S(u) \leq -y'(t)y(t)$ which implies that the solutions to (26) are stable. However, $y(t) = 0$ only contains information about the values at the boundaries; i.e., $h\left(D^{\beta}u(t,0)\right) = 0$, which does not necessarily imply $u(t,x) = 0$ for all $(t,x) \in \mathbb{R}_{\geq 0} \times \Omega$.

5. Interconnections

The next result is a small-gain theorem, which ensures stability or asymptotic stability of interconnected PDE systems under some assumptions.

Theorem 10. *Let*

$$\Sigma_i : \begin{cases} \partial_t u_i = F_i(x, D^{\alpha^i_u} u_i, D^{\alpha^i_d} d), \\ y_i = H_i(x, D^{\delta^i} u_i), \quad (t,x) \in \mathbb{R}_{\geq 0} \times \Omega \\ Q_i \begin{bmatrix} D^{\alpha^i_u - 1} u_i(t,1) \\ D^{\alpha^i_u - 1} u_i(t,0) \end{bmatrix} = 0, \qquad Q_i \begin{bmatrix} D^{\alpha^i_d - 1} d_i(t,1) \\ D^{\alpha^i_d - 1} d_i(t,0) \end{bmatrix} = 0, \end{cases} \tag{37}$$

for $i = 1, 2$. Consider the interconnected PDE systems Σ_1 and Σ_2 as depicted in Fig. 1. If Σ_1 and Σ_2 have induced $\mathcal{H}^q$-to-$\mathcal{H}^q$-norms γ_1 and γ_2, respectively, in the sense of (6), then, the interconnected system is stable in $\mathcal{H}^q_\Omega$, provided that

$$\gamma_1 \gamma_2 < 1. \tag{38}$$

Furthermore, if each of subsystems Σ_1 and Σ_2 are ZSD in $\mathcal{H}^q_\Omega$, then asymptotic stability in $\mathcal{H}^q_\Omega$ holds for the interconnected system.

Proof. Let S_1 and S_2 be two storage functionals for Σ_1 and Σ_2. By hypothesis, it holds that

$$\partial_t S_i \leq -\langle y_i, y_i \rangle_{\mathcal{H}^q_\Omega} + \gamma_i^2 \langle d_i, d_i \rangle_{\mathcal{H}^q_\Omega}, \quad i = 1, 2. \tag{39}$$

Define μ such that $\gamma_1 < \mu < \frac{1}{\gamma_2}$. Therefore, $\gamma_1\gamma_2 < \mu\gamma_2 < 1$. Let $S = S_1 + \mu^2 S_2$. Then, from (39), it follows that $\partial_t S \leq -\langle y_1, y_1 \rangle_{\mathcal{H}^q_\Omega} + \gamma_1^2 \langle d_1, d_1 \rangle_{\mathcal{H}^q_\Omega} - \mu^2 \langle y_2, y_2 \rangle_{\mathcal{H}^q_\Omega} + \mu^2 \gamma_2^2 \langle d_2, d_2 \rangle_{\mathcal{H}^q_\Omega}$. With the interconnection $y_1 = d_2$ and $y_2 = -d_1$, we have $\partial_t S \leq -(1 - \mu^2\gamma_2^2)\langle y_1, y_1 \rangle_{\mathcal{H}^q_\Omega} - (\mu^2 - \gamma_1^2)\langle y_2, y_2 \rangle_{\mathcal{H}^q_\Omega}$. Thus, from (38) and the definition of μ, it follows that the time derivative of the storage functional S is non-positive, which, in turn, implies that the interconnected PDE system is stable in $\mathcal{H}^q_\Omega$. Moreover, from ZSD property of Σ_1 and Σ_2, one can infer that $\|y_i\|_{\mathcal{H}^q_\Omega} = 0 \Rightarrow \|u_i\|_{\mathcal{H}^q_\Omega} = 0$, $i = 1, 2$. Hence, $\partial_t S(u) = 0$ only if $\|u_i\|_{\mathcal{H}^q_\Omega} = 0$, $i = 1, 2$. Consequently, from LaSalle's invariance principle, it follows that $(u_1, u_2) \to 0$ as $t \to \infty$ in $\mathcal{H}^q_\Omega$. This completes the proof.

The next corollary asserts that stability in $\mathcal{H}^q_\Omega$ holds, if both subsystems of interconnection in Fig. 1 with boundary inputs and boundary outputs have bounded $\mathcal{L}^2_{[0,\infty)}$-norms and satisfy a small gain criterion.

Corollary 11. *Let*

$$\Sigma_1 : \begin{cases} \partial_t u_1 = F_1(x, D^{\alpha_u^1} u_1) \\ y_1 = h_1(D^{\beta^1} u_1(t,1)), \\ Q_1 D^{\alpha_u^1 - 1} u_1(t,0) = w_1(t), \qquad Q_1 D^{\alpha_u^1 - 1} u_1(t,1) = 0, \end{cases}$$

and

$$\Sigma_2 : \begin{cases} \partial_t u_2 = F_2(x, D^{\alpha_u^2} u_2) \\ y_2 = h_2(D^{\beta^2} u_2(t,0)), \\ Q_2 D^{\alpha_u^2 - 1} u_2(t,0) = 0, \qquad Q_2 D^{\alpha_u^2 - 1} u_2(t,1) = w_2(t), \end{cases}$$

with interconnection $w_1 = -y_2$ and $w_2 = y_1$. If Σ_1 and Σ_2 have induced $\mathcal{L}^2_{[0,\infty)}$-norms γ_1 and γ_2, respectively, in the sense of (29), then, the interconnected system is stable in $\mathcal{H}^q_\Omega$, provided that $\gamma_1\gamma_2 < 1$.

6. Computation of storage functionals

For computational purposes, we assume that the studied PDEs are polynomial in the dependent and independent variables, i.e., functions F and H in (3) and functions F and h in (26) are all polynomials. The following structure is also considered as a candidate storage functional to check the dissipation inequalities given in Theorem 6 and Corollary 9:

$$S(u) = \frac{1}{2}\langle u, P(x)u\rangle_{\mathcal{H}^q_\Omega} := \frac{1}{2}\int_0^1 (D^q u)' P(x)(D^q u)\, \mathrm{d}x, \tag{40}$$

where, $P(x) : \Omega \to \mathbb{S}$ is a symmetric positive definite polynomial matrix function for all $x \in \Omega$. This storage functional candidate satisfies

$$\frac{1}{2}\underline{\lambda}(P)\|u\|^2_{\mathcal{H}^q_\Omega} \le S(u) \le \frac{1}{2}\bar{\lambda}(P)\|u\|^2_{\mathcal{H}^q_\Omega}. \tag{41}$$

Therefore, $(S(u))^{\frac{1}{2}}$ is equivalent to the $\mathcal{H}^q_\Omega$-norm.

6.1. PDEs with in-domain inputs/outputs

Next, we discuss how conditions of Theorem 6 are checked via integral inequalities.

Remark 12. From (41), it follows that (11) and (15) are satisfied, respectively, with $\beta_1(\cdot) = \frac{\underline{\lambda}(P)}{2}(\cdot)^2$, $\beta_1^{-1}(\cdot) = \sqrt{\frac{2}{\underline{\lambda}(P)}(\cdot)}$, and $\beta_2(\cdot) = \frac{\bar{\lambda}(P)}{2}(\cdot)^2$. □

Let $\eta = \gamma^2$. For reachability analysis, we solve the following minimization problem:

Problem 1 (*Reachability for System (3)*).

$$\text{minimize } \eta \text{ subject to (12), } \quad \text{and} \quad \nu^2 I < P(x), \tag{42}$$

where, ν is a constant.

In this case, the reachability estimate (5) transforms to

$$\|u(T,x)\|_{\mathcal{H}^q_\Omega} \le \frac{\gamma}{\nu}\|d(t,x)\|_{\mathcal{H}^p_{[0,T),\Omega}}, \quad \forall T > 0. \tag{43}$$

Similarly, for induced $\mathcal{H}^p$-to-$\mathcal{H}^q$-norm, the following minimization problem is solved:

Problem 2 (*Induced $\mathcal{H}^p$-to-$\mathcal{H}^q$ Norm for System (3)*).

$$\text{minimize } \eta \text{ subject to (14).} \tag{44}$$

When adopting the storage functional structure (40) for D^p-ISS in $\mathcal{H}^q_\Omega$, it is possible to check the condition

$$\partial_t S(u) \le -\int_\Omega (D^q u)'\alpha(x)(D^q u)\, \mathrm{d}x + \int_\Omega \sigma(|D^p d(t,x)|)\, \mathrm{d}x,$$

instead of (16), where $\alpha : \Omega \to \mathbb{S}^n$ is a symmetric positive definite polynomial function for all $x \in \Omega$. In this case, the D^p-ISS estimate translates to

$$\|u\|_{\mathcal{H}^q_\Omega} \le \left(e^{-\frac{\underline{\lambda}(\alpha)}{\bar{\lambda}(P)}t}\left(\|u_0\|^2_{\mathcal{H}^q_\Omega}\right)\right)^{\frac{1}{2}} + \left(\frac{1}{\underline{\lambda}(\alpha)}\sup_{\tau\in[0,t)}\left(\int_\Omega \sigma(|D^p d(\tau,x)|)\, \mathrm{d}x\right)\right)^{\frac{1}{2}}. \tag{45}$$

6.2. PDEs with boundary inputs and boundary outputs

In this subsection, we discuss a computational formulation of Corollary 9. To formulate the problem in terms of integral inequalities with polynomial integrands, we assume that the function σ in inequality (36) is polynomial, while the storage functional is given by (40).

Substituting (26) in inequalities (31), (33), (34), and (36) respectively yields

$$\text{(I) } \partial_t S(u) \le \left(D^{\alpha-1}u(t,1)\right)' Q' h\left(D^\beta u(t,0)\right), \tag{46}$$

$$\text{(II) } \partial_t S(u) \le \gamma^2 \left(D^{\alpha-1}u(t,1)\right)' Q' Q \left(D^{\alpha-1}u(t,1)\right), \tag{47}$$

$$\text{(III) } \partial_t S(u) \le -h'\left(D^\beta u(t,0)\right) h\left(D^\beta u(t,0)\right) + \gamma^2\left(D^{\alpha-1}u(t,1)\right)' Q' Q\left(D^{\alpha-1}u(t,1)\right), \tag{48}$$

$$\text{(IV) } \partial_t S(u) \le -\int_\Omega (D^q u)'\alpha(x)(D^q u)\, \mathrm{d}x + \sigma\left(\left|Q D^{\alpha-1}u(t,1)\right|\right), \tag{49}$$

where $\alpha : \Omega \to \mathbb{S}^n$ is a symmetric positive definite polynomial matrix function for all $x \in \Omega$. Let $\eta = \gamma^2$. For reachability analysis, the following minimization problem is solved:

Problem 3 (*Reachability for System (26)*).

$$\text{minimize } \eta \text{ subject to (47), } \quad \text{and} \quad \nu^2 I < P(x), \tag{50}$$

where, ν is a constant.

Then, the reachability estimate (28) transforms to

$$\|u(T,x)\|_{\mathcal{H}^q_\Omega} \le \frac{\gamma}{\nu}\|w(t)\|_{\mathcal{L}^2_{[0,T)}}, \quad \forall T > 0. \tag{51}$$

Analogously, we solve the following minimization problem for $\mathcal{L}^2_{[0,\infty)}$-to-$\mathcal{H}^q$-norm:

Problem 4 (*Induced $\mathcal{L}^2_{[0,\infty)}$-to-$\mathcal{H}^q$ Norm for System (26)*).

$$\text{minimize } \eta, \text{ subject to (48).} \tag{52}$$

Provided that the problem data are polynomial in the dependent variables, one can formulate convex optimization problems (SOS programs) to solve the inequalities discussed in this section. In this regard, we use the approach given in Valmorbida et al. (in press) to solve integral inequalities with polynomial integrands using convex optimization.

Table 1
Results pertained to induced $\mathcal{L}^2$-to-$\mathcal{L}^2$-norm for PDE (53).

$\frac{\sqrt{2}\beta}{\pi}$	0	0.1	0.12	0.15	0.18	0.2
γ^2	0.195	0.306	0.351	0.452	0.666	1.062

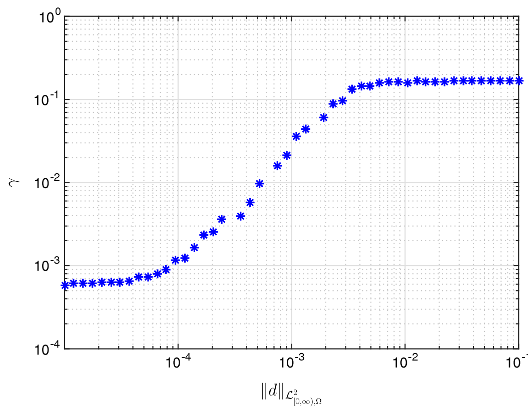

Fig. 2. The $\mathcal{L}^2$-to-$\mathcal{L}^2$ gain curve.

7. Numerical examples

In this section, we illustrate the proposed results in the paper using two numerical examples.

7.1. Example I: Reaction–diffusion–convection PDE with nonlinear forcing (Krstic, Magnis, & Vazquez, 2008; Straughan, 2004)

Consider the following PDE

$$\begin{cases}\partial_t u = \dfrac{1}{R}\partial_x^2 u - \delta u \partial_x u + \beta u^2 + d, \\ u(t,0) = 0, \qquad u(t,1) = w(t), \qquad (t,x) \in \mathbb{R}_{\geq 0} \times (0,1)\end{cases} \tag{53}$$

subject to $d(t,0) = d(t,1) = 0$ for all $t \geq 0$, where R, δ, β are constants.

7.1.1. In-domain analysis ($w \equiv 0$, $R = 1$ and $\delta = 1$)

Let $y(t,x) = u(t,x)$, $(t,x) \in \mathbb{R}_{\geq 0} \times (0,1)$. The system without inputs ($d \equiv 0$, $w \equiv 0$) is exponentially stable for $\beta < \frac{\pi}{\sqrt{2}}$ (Straughan, 2004, p. 20). Using condition (10) in Theorem 6, certificates were found for passivity just for $\beta = 0$. For the induced $\mathcal{L}^2$-to-$\mathcal{L}^2$-norm, Table 1 provides the numerical details of the numerical experiments. From numerical experiments, certificates were constructed for $\mathcal{L}^2$-norm boundedness of system (53) for $\beta \leq 0.2\frac{\pi}{\sqrt{2}}$.

At this point, let $\beta = 0$. Similar to nonlinear ODEs, we expect the nonlinear PDE to have a nonlinear induced $\mathcal{L}^2$-to-$\mathcal{L}^2$ gain function (Garulli, Masi, Valmorbida, & Zaccarian, 2013). Fig. 2 illustrates the obtained (upper bound) gain functions from numerical experiments.

7.1.2. Boundary analysis ($d \equiv 0$)

Assume $y(t) = \partial_x u(t,0)$, $t > 0$. Let $\delta = \beta = 0$. First, we study the upper bounds on γ as in (29). It is assumed $u_0(x) \equiv 0$, $\forall x \in (0,1)$. Fig. 3 illustrates the results obtained for $R \in [0.01, 10]$. For each R, Problem 4 is solved and the minimum γ is shown in the figure. As it can be inferred from the figure, as R increases and therefore the effect of the diffusion term is reduced, the obtained bounds on γ increase. At this point, we study the ISS property in $\mathcal{L}^2_\Omega$ of system (53) with $\delta = 1$ and $R = 1$. The ISS bound on β for which ISS certificates could be found was $\beta = (0.43)\frac{\pi}{\sqrt{2}}$. Fig. 4 depicts the constructed certificates $P(x)$ and $\alpha(x)$ for $\beta = (0.43)\frac{\pi}{\sqrt{2}}$. Also, the certificate $\sigma(w)$ is calculated as $\sigma(w) = 0.9506w^4 + 7.1271w^2$.

Table 2
Results pertained to induced $\mathcal{L}^2$-to-$\mathcal{H}^1$-norm for PDE (54).

$\frac{\lambda}{4\pi^2}$	0.3	0.5	0.55	0.6	0.7	0.9
γ^2	0.003	0.048	0.517	1.211	3.229	9.840

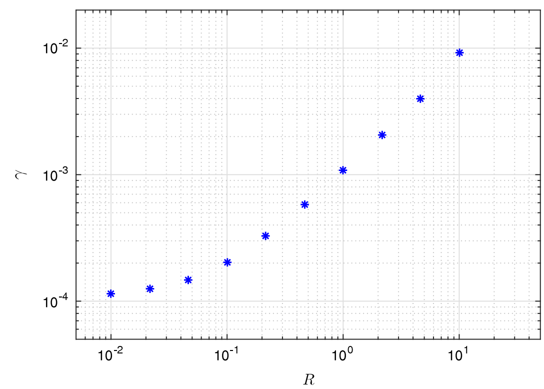

Fig. 3. The obtained upper bounds on induced $\mathcal{L}^2_{[0,\infty)}$-to-$\mathcal{L}^2$-norm.

7.2. Example II: Kuramoto–Sivashinsky equation

Consider the following PDE

$$\begin{cases}\partial_t u = -\partial_x^4 u - \lambda \partial_x^2 u - u\partial_x u + d, \\ y = u, \qquad (t,x) \in \mathbb{R}_{\geq 0} \times (0,1), \\ u(t,0) = u(t,1) = \partial_x u(t,0) = \partial_x u(t,1) = 0.\end{cases} \tag{54}$$

It was demonstrated in Lasiecka, Triggiani, Liu, and Krstić (2000) that for *constant* λ the system is exponentially stable in $\mathcal{H}^2_\Omega$-norm (thus, from Sobolev Embeddings, stable in $\mathcal{H}^1_\Omega$-norm as well) for $\lambda \leq 4\pi^2$.

First, we consider computing upper bounds on the induced $\mathcal{L}^2$-to-$\mathcal{H}^1$-norm of the system. The results are depicted in Table 2. Fig. 5 shows the elements of the 2×2 matrix $P(x)$ in the storage functional and its eigenvalues for $\lambda = (0.9)4\pi^2$. Finally, let $\lambda(x) = \lambda_0 - 16\pi^2 x(1-x)$. Then, in (54), the spatially varying coefficient $\lambda(x)$ crosses the stability bound $\lambda = 4\pi^2$ (at least) at subsets of the domain for $\lambda_0 \geq 4\pi^2$. We seek upper bounds on λ_0 such that certificates for D^1-ISS in $\mathcal{H}^1_\Omega$ can be found. For constant λ, certificates could only be found up to $\lambda = (0.62)4\pi^2$. However, for the spatially varying λ, we could construct certificates for D^1-ISS in $\mathcal{H}^1_\Omega$ for $\lambda_0 = (1.83)4\pi^2$. Fig. 6 illustrates the eigenvalues of certificates $P(x)$ and $\alpha(x)$ for the case $\lambda_0 = (1.83)4\pi^2$ and $\sigma(d, \partial_x d)$ was calculated as $\sigma(d, \partial_x d) = 3.2314d^2 + 4.0093(\partial_x d)^2$.

8. Conclusions and future work

In this paper, we proposed a methodology for the input-state/output analysis of well-posed PDEs using dissipation inequalities and we provided a systematic computational method for solving the dissipation inequalities in the case of polynomial data.

In Section 4, we delineated a method for input-state/output analysis of finite-dimensional inputs and outputs which are defined at the boundary. For results pertaining to a more general class of finite dimensional inputs and outputs and some additional discussions please refer to the addendum to this paper at Ahmadi, Valmorbida, and Papachristodoulou (2015a).

In many engineering design problems, one is interested in computing a functional of the solutions of the PDE. An example is the drag estimation problem in aerodynamics. A computational method with certificates to find bounds on such output functionals, without the need to solve the underlying PDEs, is under development (for preliminary results see Ahmadi, Valmorbida, & Papachristodoulou, 2015b).

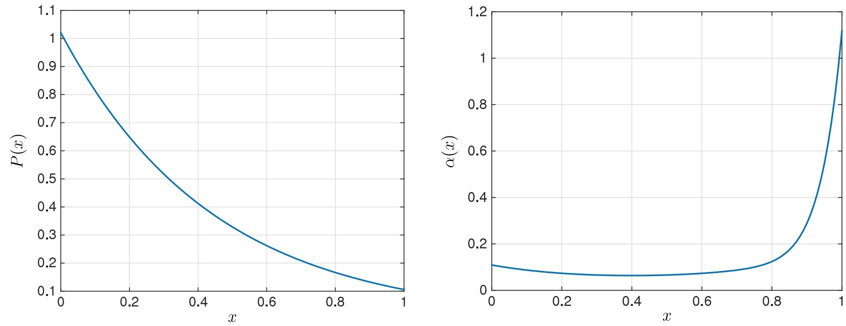

Fig. 4. The ISS certificates $P(x)$ (left) and $\alpha(x)$ (right).

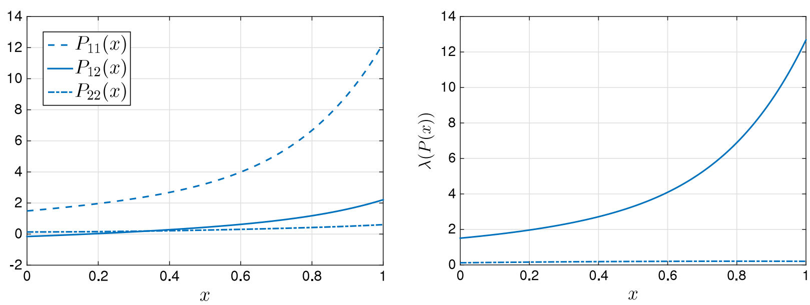

Fig. 5. The entries of $P(x)$ (left) and the eigenvalues of $P(x)$ (right) for the case $\lambda = (0.9)4\pi^2$.

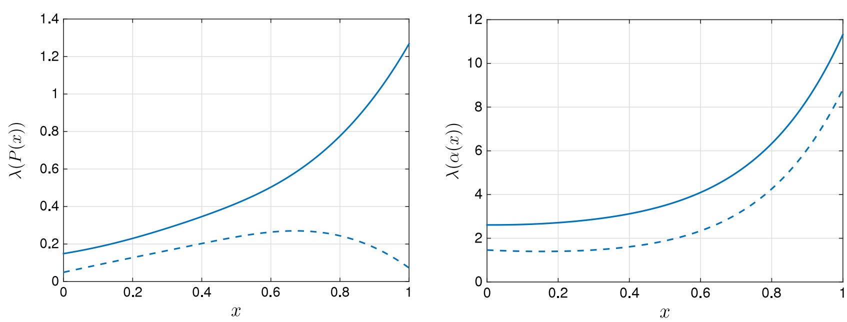

Fig. 6. D^1-ISS in $\mathcal{H}^1_\Omega$ certificates for PDE (54) with $\lambda_0 = (1.8)4\pi^2$.

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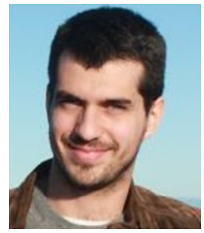

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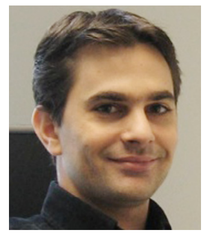

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