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Semi-definite programming and functional inequalities for Distributed Parameter Systems

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Abstract— **We study one-dimensional integral inequalities on bounded domains, with quadratic integrands. Conditions for these inequalities to hold are formulated in terms of function matrix inequalities which must hold in the domain of integration. For the case of polynomial function matrices, sufficient conditions for positivity of the matrix inequality and, therefore, for the integral inequalities are cast as semi-definite programs. The inequalities are used to study stability of linear partial differential equations.**

Keywords: Sum of Squares, Stability Analysis, Distributed Parameter Systems, PDEs.

I. INTRODUCTION

Emerging applications [1]–[5] (Magnetohydrodynamics, fluids, population dynamics) and stringent performance requirements have driven control engineering research interests towards systems described by partial differential equations (PDEs), that is, equations involving derivatives with respect to more than a single independent variable. Most commonly, the set of independent variables are temporal and spatial variables and the domain of the solutions to the PDE is unbounded for the temporal variable and bounded, in a *spatial domain*, for the spatial variable.

The study of properties of solutions to PDEs, such as stability, parallels the study of ordinary differential equations (ODEs) in several aspects. As for ODEs, conditions for stability of the zero solution can be formulated from spectral analysis when the PDE system is defined by a linear operator. Moreover, it is possible to infer stability from the semi-group generated by linear or nonlinear operators which is analogous to the ODE approach of computing solutions to establish stability of a particular solution [6]. An alternative approach relies on the Lyapunov method, extended to infinite dimensional systems in [7] and [8], which does not require the semi-groups to be calculated. The energy of the state, (which for PDEs is defined in terms of an inner product in a functional space instead of an Euclidean one), is a frequent choice for the Lyapunov functional (LF) since it simplifies the analysis of the large class of nonlinear PDE systems with energy-preserving nonlinearities [9].

Several numerical approaches for the analysis and control design of PDE systems rely on ODE approximations, obtained by spectral truncation or spatial discretization [10], [11], instead of directly addressing the PDE representation. Regarding Lyapunov stability analysis with the PDE representation, even for one-dimensional spatial domains and constant coefficients, existing analysis results rely on analytical steps [9]. These steps present increasing complexity for systems of several dependent variables, for systems with spatially varying properties (inhomogeneous systems) and for LF integrands depending on the spatial variable.

Among the numerical methods for ODEs, the ones based on semi-definite programming (SDP), a class of convex optimization problems, have been successfully applied to control problems with polynomial data. Among these problems, one can cite stability of time-delay systems [12], synthesis of polynomial control laws [13] [14], robustness analysis of polynomial systems [15] giving sum-of-squares programs (SOSP), while the primal formulation of the SOSP, the generalised problem of moments [16], has been applied to optimal control problems [17] and system analysis [18].

The connection of polynomial inequalities to semi-definite constraints is possible thanks to the non-uniqueness of quadratic-like representation of polynomials (parametrised by Gram matrices [19]). Similarly, we have non-uniqueness of integral expressions with integrands being quadratic expressions on the dependent variables and its derivatives, however, such a property has not yet been explored in the context of SDP. A hint on this direction for integral operators was reported in [20].

With the purpose of formulating numerical tests for the analysis of PDE systems, this paper studies one-dimensional integral inequalities whose integrands are *polynomial* functions of the independent spatial variables, and *quadratic* functions of the dependent variables and their spatial derivatives. The fundamental theorem of calculus (FTC) is the key step to relate the dependent variables and their derivatives in an integral expression and therefore obtain a set of representations of the integral. The matrices on the obtained quadratic expressions depend on the spatial variables and its entries are related to the boundary values of the dependent variables. This way, the positivity check of the integral on the domain is then performed by the positivity check of the matrix inequalities. We rely on the Positivstellensatz [21] in order to generate SOSPs yielding, therefore, a problem to be solved numerically.

Numerical solutions to integral inequalities are then obtained in the context of stability analysis of the $\mathcal{L}_2$ norm of systems of inhomogeneous PDEs with weighted $\mathcal{L}_2$ norm

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as LF candidates. Examples illustrating these solutions are given by: computation of bounds for the Poincaré inequalities, the stability of the heat equation with spatially varying coefficients and of the transport equation.

Notation Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ and $\mathbb{R}^n$ denote the field of reals, non-negative reals, positive reals and the n-dimensional Euclidean space respectively. The sets of natural numbers and positive natural numbers are denoted $\mathbb{N}^n$, $\mathbb{N}_0^n$. The closure of set Ω is denoted $\overline{\Omega}$. The boundary $\partial\Omega$ of set Ω is defined as $\overline{\Omega} \setminus \Omega$ with "$\setminus$" denoting set subtraction. The ring of polynomials, the ring of positive polynomials, and the ring of sum-of-squares polynomials on real variable $x \in \mathbb{R}$ are respectively denoted $\mathcal{R}[x]$, $\mathcal{P}[x]$ and $\Sigma[x]$. The ring of Sum-of-squares matrices of dimensions n is denoted $\Sigma^{n\times n}[x]$. The degree of a polynomial $p(x)$ on variable x is denoted $deg(p)$. The set of functions in a Hilbert space H on Ω are denoted $H(\Omega)$. We denote the the space of measurable functions defined on Ω as $u \in \mathcal{L}^2{}_\Omega$, the spatial $\mathcal{L}^2{}_\Omega$-norm by $\|u(t)\|_{2,\Omega} = \left(\int_\Omega u^T(t,x)u(t,x)dx\right)^{\frac{1}{2}}$ and use $\mathcal{L}^2{}_{P,\Omega}$ to denote the weighted $\mathcal{L}^2$ norm $\|u(t)\|_{(2,P),\Omega} = \left(\int_\Omega u^T(t,x)P(x)u(t,x)dx\right)^{\frac{1}{2}}$. The set of continuous functions mapping Ω into $\mathbb{R}^n$, k-times differentiable and with continuous derivatives is denoted $\mathcal{C}^k(\Omega)$. For $p \in \mathcal{C}^1(\Omega)$, the derivative of p with respect to variable x is denoted $\partial_x p$, and $\partial_x^r u := \partial_x(\partial_x^{r-1}u)$. For $u \in \mathcal{C}^k$, $\alpha \in \mathbb{N}_0^n$, define

$$D^\alpha u := (u_1, \partial_x u_1, \ldots, \partial_x^{\alpha_1} u_1, \ldots, \partial_x u_n, \ldots, \partial_x^{\alpha_n} u_n).$$

Define the *order* of $D^\alpha u$ as $ord(D^\alpha u) := \max_j \alpha_j$. We use $He(\cdot)$ to denote the linear operator $He(A) = A + A^T$. For a symmetric matrix A denote $A \geq 0$ ($A > 0$) if A is positive definite (semi-definite). The elementwise product of two vectors a, b is denoted $a \odot b$ while the elementwise inequality is denoted $a \preceq b$.

Consider $\alpha_\theta = \theta\mathbf{1_n}$, $\theta \in \mathbb{N}$, define

$$v_\theta(u(x)) := D^{\alpha_\theta} u. \tag{1}$$

The vector v_θ contains all derivatives of variable u with respect to x up to order θ. Variable u is the *dependent variable* and $x \in \Omega \subset \mathbb{R}$ the *independent variable*.

II. Positive functionals and polynomial integrands

Consider integral inequalities of the form

$$\int_\Omega \bar{f}(x, v_\theta(u))dx \geq 0, \tag{2}$$

with $\Omega = [0, 1]$, and

$$u \in \left\{ u \in \mathcal{L}_{2,\Omega} | B \begin{bmatrix} v_{\theta-1}(u(1)) \\ v_{\theta-1}(u(0)) \end{bmatrix} = 0 \right\} \tag{3}$$

with $B \in \mathbb{R}^{n_b \times 2n(\theta-1)}$.

Assume that $\bar{f}$ is quadratic on the second argument for any value of the first argument, that is

$$\bar{f}(x, v_\theta(u)) = v_\theta^T(u)F(x)v_\theta(u). \tag{4}$$

It is further assumed that $F(x) \in \mathcal{C}^0(\Omega)$.

The remaining of this section aims to derive conditions for (2) to hold in terms of expressions involving only $F(x)$ in the integrand (4) and the boundary values of the dependent variable as defined in (3). To this aim, the following result is fundamental.

Lemma 1: Consider $r : \Omega \to \mathbb{R}^{n_r}$, $r \in \mathcal{C}^1$. If there exists a vector function $h : \Omega \to \mathbb{R}^{n_r}$, $h \in \mathcal{C}^1$ satisfying $h(x) \odot r(u(x)) \preceq 0$ for $x \in \partial\Omega$, then

$$\int_\Omega \left[\frac{d}{dx} h(x) \odot r(x) + h(x) \odot \frac{d}{dx} r(x) \right] dx \leq 0 \tag{5}$$

Proof: From the fundamental theorem of calculus, one has

$$\begin{aligned} h(x) \odot r(x)|_{\partial\Omega} &= \int_\Omega \left[\frac{d}{dx}(h(x) \odot r(x)) \right] dx \\ &= \int_\Omega \left[\frac{d}{dx} h(x) \odot r(x) + h(x) \odot \frac{d}{dx} r(x) \right] dx, \end{aligned}$$

since $h(x) \odot r(x) \preceq 0$ for $x \in \partial\Omega$, one obtains (5). ■

Whenever $r(x)$ is a vector of monomials on the entries of $v_\theta(u(x))$, the integrand in (5) relates the monomials *explicitly* accounting for the dependence of u on variable x as follows.

Corollary 1: Consider $v_{\theta-1}^{\{2\}}(u)$, the vector containing all monomials of degree 2 on $v_{\theta-1}$, and the set

$$\mathcal{H}(\theta) := \left\{ h \in \mathcal{C}^1(\Omega) : h(x) \odot v_{\theta-1}^{\{2\}}(u)|_{\partial\Omega} \preceq 0 \right\}. \tag{6}$$

If $h \in \mathcal{H}(\theta)$, then

$$\int_\Omega \frac{d}{dx} h(x) \odot v_{\theta-1}^{\{2\}}(u) + h(x) \odot C v_\theta^{\{2\}}(u) dx \preceq 0, \tag{7}$$

where C is the matrix satisfying $\frac{d}{dx} v_{\theta-1}^{\{2\}}(u) = C v_\theta^{\{2\}}(u)$.

The corollary is proven by considering $r(x) = v_\theta^{\{2\}}$ in (5).

Example 1 Consider $\Omega = [0, 1]$, $u = u_1$, that is, $n = 1$ and take $\theta = 2$. The set in (6), is defined with $v_\theta^{\{2\}} = (u^2(x), u(x)\partial_x u(x), (\partial_x u(x))^2)$. Consider $u(t, 0) = u(t, 1) = 0$. The hypothesis of Corollary 1 holds with $h(x) = (h_1(x), h_2(x), h_3(x))$ satisfying $h_3(0) \leq 0$ and $h_3(1) \leq 0$ and arbitrary values for h_1 and h_2 at the boundaries since $u^2(t, 1) = u^2(t, 0) = u(t, 1)\partial_x u(t, 1) = u(t, 0)\partial_x u(t, 0) = 0$. If instead the values at the boundaries are given by $u(t, 0) = u(t, 1)$, $\partial_x u(t, 0) = \partial_x u(t, 1)$, the hypothesis is satisfied with $h_1(1) - h_1(0) \leq 0$, $h_2(1) - h_2(0) = 0$ and $h_3(1) - h_3(0) \leq 0$.

Define $\bar{h}$ as the integrand in, that is (7) $\bar{h} := \frac{d}{dx} h(x) \odot v_{\theta-1}^{\{2\}}(u) + h(x) \odot C v_\theta^{\{2\}}(u)$ which is a vector of n_r elements, quadratic on the dependent variables v_θ. We can then write

$$\textstyle\sum_i^{n_r} \bar{h}(x, v_\theta(u)) \quad = \quad v_\theta^T H(x) v_\theta, \tag{8}$$

with $\bar{k} = \left\lceil \frac{k}{2} \right\rceil$.

Example 2 Consider $h(x)$ and $v_\theta^{\{2\}}$ as in Example 1, then

$$\bar{h} = \begin{bmatrix} \frac{dh_1}{dx} u^2 + 2h_1 u^2 \partial_x u \\ \frac{dh_2}{dx} u\partial_x u + h_2(u\partial_{xx} u + (\partial_x u)^2) \\ \frac{dh_3}{dx} (\partial_x u)^2 + 2h_3 u\partial_x u \end{bmatrix}$$

the matrix $H(x)$ in (8) is given by

$$H(x) = \begin{bmatrix} \frac{dh_1}{dx} & h_1 + \frac{1}{2}\frac{dh_2}{dx} & \frac{1}{2}h_2 \\ h_1 + \frac{1}{2}\frac{dh_2}{dx} & h_2 + \frac{dh_3}{dx} & h_3 \\ \frac{1}{2}h_2 & h_3 & 0 \end{bmatrix}.$$

Remark 1: Recall that, from the definition of $\mathcal{H}(\theta)$, information about the values of the dependent variables at the boundaries define the values at the boundary of the entries of $H(x)$. $\star$

Proposition 1: If $\exists h \in \mathcal{H}(\theta)$, such that

$$T(x) := F(x) + H(x) \geq 0 \quad \forall x \in \Omega \tag{9}$$

with $F(x)$ and $H(x)$ respectively as in (4) and in (8), then inequality (2) holds.

Proof: Consider $h \in \mathcal{H}(\theta)$ such that $T(x) \geq 0$ then

$$\begin{aligned} 0 &\leq \int_\Omega v_\theta^T T(x) v_\theta dx \\ &= \int_\Omega v_\theta^T \left[F(x) + H(x)\right] v_\theta dx \\ &= \int_\Omega v_\theta^T F(x) v_\theta dx + \int_\Omega v_\theta^T H(x) v_\theta dx \\ &= \int_\Omega \bar{f}(x, v_\theta(u)) dx + \int_\Omega \sum_{i=1}^{n_r} \bar{h}_i(x, v_\theta(u)) dx \\ &\leq \int_\Omega \bar{f}(x, v_\theta(u)) dx. \end{aligned} \tag{10}$$

■

Remark 2: Since the elements of $H(x)$ involve continuously differentiable functions and their derivatives, (9) is a differential matrix inequality. If we further assume that the functions h and f are polynomials on x, it is possible to formulate convex feasibility problem to solve (9) as presented in the next section. $\star$

III. Positivity in the domain

The case of $T(x)$ in (9) being a polynomial on variable x is addressed in this section. The following result is a straightforward application of the Putinar's Positivstellensatz (see Theorem 2 in the appendix) to (9), to hold in the set $\Omega = [0, 1]$, characterized as the semi-algebraic set $\{x | x(1-x) \geq 0\}$.

Corollary 2: If there exists $N(x) \in \Sigma^{n_M \times n_M}[x]$ such that

$$T(x) - N(x)x(1-x) \in \Sigma^{n_M \times n_M}[x] \tag{11}$$

then (9) holds.

Remark 3: Whenever $T(x)$ and $N(x)$ are unknown with affine in a set of variables, (for instance when coefficients of $\bar{f}$ and h are set as variables), the above test can be formulated as a SDP whose dimension depends on the degree of the polynomials in variables x. $\star$

IV. Stability Analysis for Distributed Parameter Systems

We consider the following PDE system

$$\partial_t u = \mathcal{A}u, \quad u(x, 0) = u_0(x) \in \mathcal{M} \subset H(\Omega) \tag{12}$$

where $H(\Omega)$ is an infinite-dimensional Hilbert space and $\mathcal{A}$ is a linear operator defined on $\mathcal{M}$, a closed subset of $H(\Omega)$. It is assumed that $\mathcal{A}$ generates a linear semi-group of contractions, *i.e.* continuous solutions to the PDE exist in $\mathcal{M}$ and are unique [6].

Consider candidate Lyapunov functions of the form

$$V(u) = \frac{1}{2}\|u\|_{2,P}^2 = \frac{1}{2}\int_\Omega u^T P(x) u \; dx, \;\; P(x) > 0 \; \forall x \in \Omega, \tag{13}$$

for fixed $t > t_0$. The following lemma states the equivalence of the weighted norm and the $\mathcal{L}^2$-norm.

Lemma 2: If $P(x) > 0 \quad \forall x \in \bar{\Omega}$ then the norms $\|u\|_{2,P(x)}$ and $\|u\|_2$ are equivalent.

Proof: Let $\lambda_M(P, \Omega) := \max_{\bar{\Omega}}(\lambda(P(x)))$, $\lambda_m(P, \Omega) = \min_{\bar{\Omega}}(\lambda(P(x)))$. One has

$$\begin{aligned} \|u\|_{2,P(x)}^2 &= \left[\int_\Omega u^T P(x) u \; dx\right] \\ &\leq \lambda_M(P, \Omega)\left[\int_\Omega u^T u \; dx\right] = \lambda_M(P, \Omega)\|u\|_2^2 \end{aligned} \tag{14}$$

$$\begin{aligned} \|u\|_{2,P(x)}^2 &= \left[\int_\Omega u^T P(x) u \; dx\right] \\ &\geq \lambda_m(P, \Omega)\left[\int_\Omega u^T u \; dx\right] = \lambda_m(P, \Omega)\|u\|_2^2. \end{aligned} \tag{15}$$

Therefore

$$\sqrt{\lambda_m(P, \Omega)}\|u\|_2 \leq \|u\|_{2,P(x)} \leq \sqrt{\lambda_M(P, \Omega)}\|u\|_2.$$

■

The following proposition is a Lyapunov result for the exponential convergence of the $\mathcal{L}^2$ norm of the solutions to (12):

Theorem 1: Suppose there exists a function V is a functional $V(0) = 0$, and scalars c_1, c_2, $c_3 \in \mathbb{R}_{>0}$ such that

$$c_1\|u\|_{2,\Omega} \leq V(u) \leq c_2\|u\|_{2,\Omega} \tag{16}$$

$$\frac{dV}{dt}(u) \leq -c_3\|u\|_{2,\Omega} \tag{17}$$

then the $\mathcal{L}^2$ norm of the trajectories of (12) satisfy

$$\|u(t, x)\|_{2,\Omega} \leq \frac{c_2}{c_1}\|u(t_0, x)\|_{2,\Omega} e^{-\frac{c_3}{c_1}(t - t_0)}. \tag{18}$$

Proof: From (16)-(17) one obtains

$$\frac{\frac{dV}{dt}(u)}{V(u)} \leq -\frac{c_3}{c_1}$$

since $\frac{\frac{dV}{dt}(u)}{V(u)} = \frac{d(\ln(V(u)))}{dt}$, the integral of the above expression over $[t_0, t]$, gives

$$\int_{[t_0,t]} \frac{d(\ln(V(u(\tau, x))))}{d\tau} d\tau \leq -\frac{c_3}{c_1}(t - t_0)$$

$$\ln(V(u(t, x))) - \ln(V(u(t_0, x))) \leq -\frac{c_3}{c_1}(t - t_0)$$

$$\frac{V(u(t, x))}{V(u(t_0, x))} \leq e^{-\frac{c_3}{c_1}(t - t_0)}$$

$$V(u(t, x)) \leq V(u(t_0, x)) e^{-\frac{c_3}{c_1}(t - t_0)}$$

finally (18) is obtained by applying the bounds of (16) on the above inequality. ■

Corollary 3: If there exists a symmetric matrix $P(x)$ and positive scalars ϵ_1, ϵ_2 such that, for t held fixed

$$\frac{1}{2}\int_\Omega \left(u^T P(x)u - \epsilon_1 u^T u\right)\ dx \geq 0, \tag{19}$$

$$-\int_\Omega \left(u^T P(x)\mathcal{A}u + \epsilon_2 u^T u\right)\ dx \geq 0. \tag{20}$$

Then the $\mathcal{L}^2$ norm of solutions to (12) satisfy (18).

V. EXAMPLES

In this section, we obtain solutions to the integral inequalities using the formulation developed in Sections II and III.

A. *Poincaré inequality*

The Poincaré inequality [23, p.163]

$$\int_\Omega u^2 dx \leq \kappa \int_\Omega (\partial_x u)^2 dx \tag{21}$$

$u(t,0) = u(t,1) = 0$ where Ω is a bounded domain and κ is a constant depending on the domain, holds for all $u, \partial_x u \in \mathcal{L}^2{}_\Omega$ and establishes bounds for $\|u\|_2^2$ in terms of $\|\partial_x u\|_2^2$. By rewriting the above inequality as

$$\int_\Omega \left(\kappa(\partial_x u)^2 - u^2\right) dx \geq 0 \tag{22}$$

one obtains an integral constraint of the form (2). Notice that the integrand is affine on κ. Tight bounds for (21) are obtained by solving

$$\begin{array}{ll} \text{minimize } \kappa & \\ \text{subject to } \int_\Omega \left(\kappa(\partial_x u)^2 - u^2\right) dx \geq 0. \end{array} \tag{23}$$

The steps described in Section II are followed by first noticing that the integrand of (22) involves only u and its spatial derivative $\partial_x u$, therefore let $\theta = 1$ in (7) and $v_{\theta-1}(u) = u^2$. Following Proposition 1, the problem (23) becomes

$$\begin{array}{ll} \text{minimize } \kappa & \\ \text{subject to } \begin{bmatrix} -1 + \frac{dh(x)}{dx} & h(x) \\ h(x) & \kappa \end{bmatrix} \geq 0, \forall x \in \Omega. \end{array} \tag{24}$$

Assuming $h(x)$ to be of polynomial form, $\Omega = [0,1]$ and applying Positivstellensatz as described in Section III, (24) becomes the following SOSP

$$\begin{array}{ll} \text{minimize } \kappa & \\ \text{subject to} & \begin{bmatrix} -1 + \dfrac{dh(x)}{dx} & h(x) \\ h(x) & \kappa \end{bmatrix} \\ & \qquad + N(x)x(x-1) \in \Sigma^{2\times 2}[x], \\ & N(x) \in \Sigma^{2\times 2}[x]. \end{array} \tag{25}$$

The problem (25) is formulated and solved using SOSTOOLS [24] considering different degrees for polynomial $h(x)$ and $N(x)$. Figure 1 depicts the optimal value κ^* as a function of the degree of $h(x)$ (the curve was computed setting $deg(N(x)) = deg(h(x))+2$). The figure also presents the optimal bound π^{-2} for the studied domain [25].

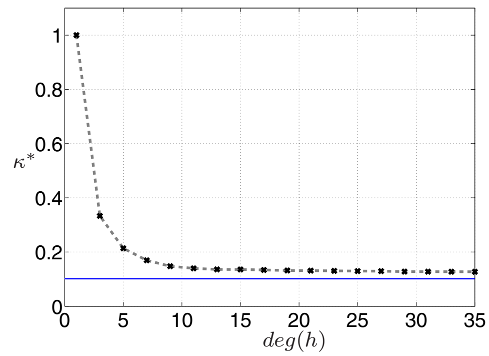

Fig. 1. Optimal values for problem (25) as a function of the degree of $h(x)$.

B. *The transport equation*

Consider the following PDE

$$\partial_t u = -\partial_x u \quad x \in [0,1],\ t > 0 \quad u(t,0) = 0.$$

Let $E_p := \dfrac{1}{2}\int_\Omega e^{-\lambda x} u^2 dx$ be the candidate function to certify $-\lambda E_p - \frac{dE_p}{dt} \geq 0$ that is, to certify exponential stability with exponential rate $\lambda > 0$. One has

$$-\lambda E_p - \frac{dE_p}{dt} = \int_\Omega \left(-\frac{\lambda}{2}e^{-\lambda x}u^2 + e^{-\lambda x}u\partial_x u\right) dx \geq 0, \tag{26}$$

which is an inequality as (2). Consider $\eta_2(v_1(u)) = u^2$ and $h(x) = -\frac{1}{2}e^{-\lambda x}$. Since $h(1) = -\frac{1}{2}e^{-\lambda} < 0$, one has $h(1)u^2(1) - h(0)u^2(0) = h(1)u^2(1) \leq 0$, hence $h(x) \in \mathcal{H}(1)$ and

$$\begin{aligned} h(x)\eta_2(v_1(u))|_{\partial\Omega} &= h(1)u^2(t,1) \\ &= \int_\Omega \left(h_x u^2 + hu\partial_x u\right) dx \\ &= \int_\Omega \left(\frac{1}{2}\lambda e^{-\lambda x}u^2 - e^{-\lambda x}u\partial_x u\right) dx \leq 0 \end{aligned}$$

where equality holds only if $u(t,1) = 0$. Adding up $-\lambda E_p - \frac{dE_p}{dt}$ and $h(1)u^2(t,1)$ one obtains

$$\begin{aligned} &-\lambda E_p - \frac{dE_p}{dt} + h(1)u^2(t,1) \\ &\quad = \int_\Omega \left(-\frac{\lambda}{2}e^{-\lambda x}u^2 + e^{-\lambda x}u\partial_x u\right) dx \\ &\qquad + \int_\Omega \left(\frac{\lambda}{2}e^{-\lambda x}u^2 - e^{-\lambda x}u\partial_x u\right) dx = 0 \end{aligned}$$

therefore

$$-\lambda E_p - \frac{dE_p}{dt} = -h(1)u^2(t,1),$$

since $-h(1)u^2(t,1) \leq 0$, exponential stability of the zero solution is proven for *any* convergence rate $\lambda > 0$. This result should be expected as, for a compact and bounded domain, the transport equation is *finite-time* stable.

By considering inequalities (19)-(20) with a polynomial weighting function and considering polynomial $h(x) \in$

$\mathcal{H}(1)$, the Positivstellensatz is applied to formulate the following feasibility SOSP

$$\begin{aligned}
&\text{find } p(x),\ h(x),\ N(x) \\
&\text{subject to} \\
&He\left(\frac{1}{2}\begin{bmatrix} -\lambda p(x) + \frac{dh(x)}{dx} & -p(x)+h(x) \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad\qquad + N(x)x(x-1) \in \Sigma^{2\times 2}[x], \\
&N(x) \in \Sigma^{2\times 2}[x].
\end{aligned} \tag{27}$$

With a polynomial $p(x)$ of degree 30 stability of the zero solution to (26) was certified for $\lambda \in (0, 10]$. The results are depicted in Figure 2.

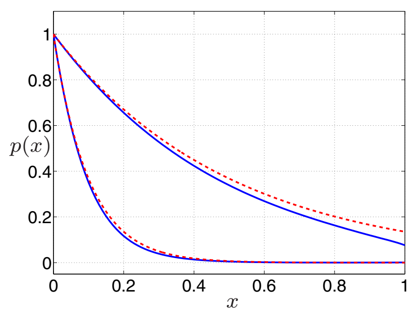

Fig. 2. Weighting functions proving exponential stability for convergence rates $\lambda \in \{2, 10\}$. The red dotted curves depict the analytical result $\frac{1}{2}e^{-\lambda x}$ while the solid blue lines are correspond to the polynomials obtained by solving (27).

C. Heat Equation with Reaction Term

Consider the following inhomogeneous PDE

$$\partial_t u = \partial_x^2 u + \lambda(x)u,\ x \in [0,1],\ u(t,0) = u(t,1) = 0 \tag{28}$$

where, $\lambda : [0,1] \to \mathbb{R}$. When $\lambda(x) = \lambda_c$, the Lyapunov functional $\int_0^1 u^2\, dx$, proves asymptotic stability for $\lambda_c \in (-\infty, \pi^2)$. In order to study the exponential stability of (28), we consider a weighted $\mathcal{L}^2$ function as (13).

In [20] the system was studied with $\lambda(x) = \lambda_c$ and employing an *ad hoc* integration by parts construction to obtain a tight estimate for the stability interval. Here $\lambda(x)$ is considered as $\lambda(x) = \lambda_c - 24x + 24x^2$ and a line search was performed to maximize the coefficient λ_c for which the zero solution is stable. We obtained the value $\lambda_c^* = 14.1$ by solving (19)-(20) with a polynomial weighting function. Figure 3 depicts $\lambda(x)$ with $\lambda_c = \lambda_c^*$. The stability bound for a constant coefficient $\lambda(x) = \pi^2$, is also depicted. Notice that $\lambda(x) > \pi^2$ for $x \in [0, 0.2209) \cup (0.7791, 1]$. The obtained weighting function $p(x)$, a polynomial of degree 10 is illustrated in Figure 4.

VI. Conclusion

This paper has formulated conditions for the positivity of integral inequalities in terms of positivity of their integrands by characterizing a set of expressions constructed from the Fundamental Theorem of Calculus. The main assumption is that the functionals under study are quadratic on the

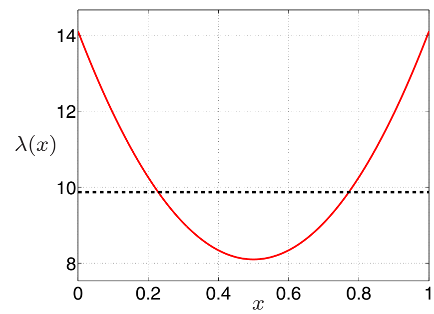

Fig. 3. The spatially varying coefficients $\lambda = \pi^2$ (dashed black) $\lambda(x) = \lambda_c - 24x + 24x^2$ (solid red).

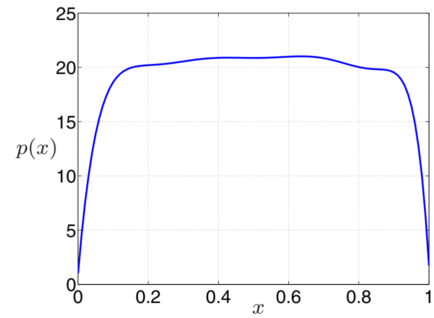

Fig. 4. The weighting function $p(x)$ for system (28).

dependent variables. The case of polynomial dependence of the integrand on the independent variable allows for the formulation of a convex optimization problem given by SDPs.

These formulations were then used to study integral inequalities arising from Lyapunov stability conditions for PDEs and examples illustrate the effectiveness of the proposed approach.

Polynomial parametrization of the weighting functions on the Lyapunov functionals is not restrictive since, according to Weierstrass approximation theorem, any continuous function on a bounded interval can be approximated by a polynomial, as illustrated by Figure 2 in Example V-B. The drawback is that the degree of the approximating polynomial may not be known *a priori*. The main source of conservativeness of the presented conditions is related to the fact that we only consider weighted energy as linear functionals.

The research leading to the results presented here was motivated from the fact that integration by parts is a crucial step when proving stability analytically. Another important step is accounting for the size of the domain which is usualy performed by considering embedding theorems on bounded domains. Our goal is to perform these steps by solving semi-definite programs.

We believe the results presented in Sections II and III go beyond the scope of stability analysis of PDEs, and provide an efficient method of formulating a set of optimization problems with integral constraints in a convex optimization framework.

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APPENDIX

A. Sum-of-Squares Polynomials

A polynomial $p(x)$ is a sum-of-squares (SOS) polynomial if $\exists p_i(x) \in \mathcal{R}[x]$, $i \in \{1,\dots,n_d\}$ such that $p(x) = \sum_i p_i^2(x)$. Hence $p(x)$ is clearly non-negative. A set of polynomials p_i satisfying the above property is called *SOS decomposition* of $p(x)$. The converse does not hold in general, that is, there exist non-negative polynomials which do not have an SOS decomposition [26]. The computation of SOS decompositions, can be cast as an SDP (see [19], [26], [27]). The theorem below proves that, in sets satisfying a property slightly stronger than compactness, any positive polynomial can be expressed as a combination of sum-of-squares polynomials and polynomials describing the set.

For a set of polynomials $\bar{g} = \{g_1(x),\dots,g_m(x)\}$, $m \in \mathbb{N}$, the *quadratic module* generated by m is

$$M(\bar{g}) := \left\{ \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i | \sigma_i \in \Sigma[x] \right\}. \tag{29}$$

A quadratic module $M \in \mathcal{R}[x]$ is said *archimedean* if $\exists N \in \mathbb{N}$ such that

$$N - \|x\|_2^2 \in M.$$

An archimedian set is always compact [28]. It is the possible to state [16, Theorem 2.14]

Theorem 2 (Putinar Positivstellensatz): Suppose the quadratic module $M(\bar{g})$ is Archimedian. Then for every $f \in \mathcal{R}[x]$,

$$f > 0 \;\forall\; x \in \{x | g_1(x) \geq 0,\dots,g_m(x) \geq 0\} \Rightarrow f \in (\bar{g}).$$

Lemma 3: The set $\Omega = [0,1]$ is Archimedean.

Take any pair (r, N^*), $r \in \mathbb{R}_{>0}$ and $N^* \in \mathbb{N}$ satisfying

$$N^* \geq \frac{1}{4}\frac{r^2}{r-1}. \tag{30}$$

$r \neq 1$. The Archimedean property is then satisfied with

$$\begin{aligned} \theta_0(\sigma) &= \left(\left(\sqrt{r-1}\right)\sigma - \tfrac{1}{2}\tfrac{r}{\sqrt{r-1}} \right)^2 \\ &\quad + \left(N^* - \tfrac{1}{4}\tfrac{r^2}{(r-1)} \right) \\ \theta_1(\sigma) &= r. \end{aligned}$$