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Convex Solutions to Integral Inequalities in Two-Dimensional Domains

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Abstract— **This paper presents a method to verify integral inequalities on two-dimensional domains. The integral expressions are given by line integrals on the boundaries and by surface integrals: both are quadratic on the dependent variables and their derivatives. The proposed approach can verify the inequalities for a set of the dependent variables defined by their boundary values. We apply the results to solve integral inequalities related to Lyapunov stability conditions for exponential stability of Partial Differential Equations.**

I. Introduction

Several engineering systems model are described by Partial Differential Equations (PDEs), obtained from mass and energy conservation laws [1], [2]. Most commonly the independent variables on these models are time and spatial variables. In this paper we are interested in systems defined in spatial domains of two dimensions.

Properties of 2D PDE models such as stability of solutions or input-output gains may be difficult to extract. Obviously, simulations can not provide certificates of these properties. A possible approach to the analysis relies on approximations of PDEs, obtained via modal decomposition, and on the use of tools developed for Ordinary Differential Equations (ODEs) [3], [4]. However, it may be difficult to evaluate whether the approximation is a reliable model and whether the computed properties of the approximation hold for the original PDE model.

Among the challenging problems related to two-dimensional PDE models we can list a set of fluid flow analysis problems. In [5], drag reduction properties of viscous, non-Newtonian flows were studied. The amplification properties of the linearized Navier Stokes (NS) equation is considered in [6] for stochastically excited equations. These 2D fluid flow models are simplifications of the three dimensional Navier-Stokes equations under assumptions of invariance in one direction. A destabilizing boundary control is designed to enhance the mixing pattern of a pressure-driven control in [7]. Another well-studied problem in two-dimensional domain is given by heat diffusion processes. In [8], the computation of a boundary feedback law is proposed for a pool-boiling system actuated from the bottom of the pool.

In the context of stability analysis, Lyapunov's second method was extended to study stability of PDE systems in [9], [10]. One possible class of Lyapunov functional (LF) candidates for PDE systems is the weighted H^q-norm. These functionals yield Lyapunov inequalities that are integral inequalities, which may then be solved analytically or numerically.

This paper presents a method to verify two-dimensional integral inequalities with integrands given by quadratic expressions in the dependent variables. Conditions for the integrals to hold are formulated in terms of the matrices describing the quadratic integrands. The proposed method is then applied to study the stability of PDEs in two dimensional domains, using weighted $\mathcal{L}_2$ norms as LF candidates. For integrands which are polynomial in the *independent* variables, the conditions for the integral inequalities to hold become polynomial matrix inequalities. Such a structure allows us to exploit the semidefinite programming (SDP) formulation [11] of optimization problems with linear objective functions and polynomial constraints.

The paper is organised as follows. Section II defines the problem and describes a method to verify integral inequalities with quadratic integrands by studying matrix inequalities. In Section III, for domains given by the unit square, we detail an equivalent formulation to a line integral inequality given by a one-dimensional inequality. In Section IV, some results for the stability analysis of PDE systems are summarised. Examples illustrate the application of the proposed method to the stability analysis of PDEs in Section V. Finally, Section VI concludes the paper and gives directions for future research.

The set of continuous vector functions, mapping $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^n$, which are k-times differentiable and have continuous derivatives is denoted $\mathcal{C}^k(\Omega)$. The set of k-times differentiable continuous functions with continuous derivatives with respect to variable θ is denoted $\mathcal{C}^k(\Omega,\theta)$. For $p(x,y) \in \mathcal{C}^1(\Omega,\theta)$, the derivative of p with respect to variable θ is denoted $\partial_\theta p$. For $u \in \mathcal{C}^\alpha(\Omega)$, $\alpha \in \mathbb{N}_0$, define $D^\alpha u := \left(u, \partial_x u, \partial_y u, \partial_x^2 u, \partial_{xy} u, \partial_y^2 u, \ldots, \partial_y^\alpha u\right)$, α is the *order* of $D^\alpha u$. For $u \in \mathcal{C}^\alpha(\Omega,\theta)$, $\alpha \in \mathbb{N}_0$, define $D_\theta^\alpha u := (u, \partial_\theta u, \partial_\theta^2 u, \ldots, \partial_\theta^\alpha u)$. Define $\sigma(r,k) := \binom{r+k-1}{r-1} = \frac{(r+k-1)!}{(r-1)!k!}$. For $u : \Omega \to \mathbb{R}^n$, $u \in \mathcal{C}^\alpha$, $\Omega \subset \mathbb{R}^2$ we have $D^\alpha u : \Omega \to \mathbb{R}^{n\sigma(2,\alpha)}$. The variable u is called the *dependent variable* and x, y, the *independent variables*. For notational convenience, depending on the context we may suppress the dependence of $u(t,x,y)$ on t (by writing $u(x,y)$) or (x,y) (by writing $u(t)$) or both (by writing u). The set of functions satisfying $\|u\|_\alpha^2 = \int_\Omega (D^\alpha u(x,y))^T (D^\alpha u(x,y))\,\mathrm{d}\Omega < \infty$ is denoted H_Ω^α. The set of square integrable functions $\mathcal{L}_\Omega^2$ corresponds to H^0. For a matrix $P(x,y) >$

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$0 \ \forall x \in \Omega$, define the norm $\|u\|_{(\alpha,P(x,y))} := \left(\int_\Omega (D^\alpha u(x,y))^T P(x,y)(D^\alpha u(x,y)) \, \mathrm{d}\Omega\right)^{\frac{1}{2}}$. The set of sum of squares polynomial matrices on real variables x and y of dimension $N \times N$ is denoted $\Sigma^{N\times N}[x,y]$. The set of real symmetric matrices is denoted $\mathbb{S}^n = \{A \in \mathbb{R}^{n\times n} | A = A^T\}$. For $A \in \mathbb{S}^n$, denote $A \geq 0$ ($A > 0$) if A is positive semidefinite (definite). The linear operator $He(\cdot)$ satisfies $He(A) := A + A^T$. $Diag(A,B)$ denotes the block-diagonal matrix formed by matrices A and B.

II. Integral inequalities with quadratic integrands in two dimensions

In this section, we study inequalities defined in two-dimensional domains. These inequalities are composed of surface integrals in a two-dimensional domain and line integrals on the boundary of the domain. In both integrals, the integrand is quadratic on the dependent variables.

Consider the integral inequality

$$\oint_{\partial\Omega} \begin{bmatrix} \left(D^{\alpha-1}u(x,y)\right)^T M_{b1}(x,y) D^{\alpha-1}u(x,y) \\ \left(D^{\alpha-1}u(x,y)\right)^T M_{b2}(x,y) D^{\alpha-1}u(x,y) \end{bmatrix} \cdot \mathrm{d}\ell + \int_\Omega (D^\alpha u(x,y))^T M_i(x,y) D^\alpha u(x,y) \, \mathrm{d}\Omega \geq 0, \quad (1)$$

$M_{bk} : \Omega \to \mathbb{S}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha-1)}$, $k \in \{1,2\}$, $M_i : \Omega \to \mathbb{S}^{n\sigma(2,\alpha)\times n\sigma(2,\alpha)}$ with the dependent variable u in the set

$$\mathcal{B} := \left\{ u(x,y) \in \mathcal{H}^\alpha_\Omega \middle| f(D^{\alpha-1}u(x,y)) = g(x,y) \right\}. \quad (2)$$

with $f : \mathbb{R}^{n\sigma(2,\alpha-1)} \to \mathbb{R}^{n_b}$, $g : \partial\Omega \to \mathbb{R}^{n_b}$.

We study the following problem:

Problem 1: Check whether the integral inequality (1) holds for all $u(x,y) \in H^\alpha(\Omega)$ satisfying $u(x,y) \in \mathcal{B}$.

Green's Theorem generalises the Fundamental Theorem of Calculus (FTC) for two dimensions. The integrands of the surface and the line integrals in (1) are respectively defined as quadratic forms of vectors $D^\alpha u$ and $D^{\alpha-1}u$. For quadratic integrands, Green's Theorem yields the following result:

Lemma 1 (Green's Theorem - quadratic forms): For $H_i : \mathbb{R}^2 \to \mathbb{R}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha-1)}$, $H_i \in \mathcal{C}^1(\Omega)$, $i = 1,2$ and $u \in \mathcal{C}^\alpha(\Omega)$, the following holds

$$\oint_{\partial\Omega} \begin{bmatrix} (D^{\alpha-1}u(x,y))^T H_1(x,y) D^{\alpha-1}u(x,y) \\ (D^{\alpha-1}u(x,y))^T H_2(x,y) D^{\alpha-1}u(x,y) \end{bmatrix} \cdot \mathrm{d}\ell - \int_\Omega \left((D^\alpha u(x,y))^T \bar{H}(x,y) \left(D^\alpha u(x,y)\right)\right) \mathrm{d}\Omega = 0, \quad (3)$$

where

$$\bar{H}(x,y) := \begin{bmatrix} -\partial_y H_1(x,y) + \partial_x H_2(x,y) & 0 \\ 0 & 0 \end{bmatrix} + He\left(\begin{bmatrix} -H_1(x,y) \\ 0 \end{bmatrix} M_{\partial_y} + \begin{bmatrix} H_2(x,y) \\ 0 \end{bmatrix} M_{\partial_x} \right)$$

where matrices $M_{\partial_x} \in \mathbb{N}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha)}$, $M_{\partial_y} \in \mathbb{N}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha)}$ are permutation matrices satisfying $\partial_x \left(D^{\alpha-1}u(x,y)\right) = M_{\partial_x} D^\alpha u(x,y)$, $\partial_y \left(D^{\alpha-1}u(x,y)\right) = M_{\partial_y} D^\alpha u(x,y)$.

We can then state our main result which provides a condition to verify the integral inequality (1).

Theorem 1: If there exist matrix functions $H_i : \Omega \to \mathbb{S}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha-1)}$, $H_1 \in \mathcal{C}^1(\Omega, y)$, $H_2 \in \mathcal{C}^1(\Omega, x)$, such that

$$\oint_{\partial\Omega} \begin{bmatrix} \left(D^{\alpha-1}u\right)^T (M_{b1} + H_1) D^{\alpha-1}u \\ \left(D^{\alpha-1}u\right)^T (M_{b2} + H_2) D^{\alpha-1}u \end{bmatrix} \cdot \mathrm{d}\ell \geq 0 \quad (6)$$

$\forall u \in \mathcal{B}$ and

$$M_i(x,y) - \bar{H}(x,y) \geq 0, \ \forall (x,y) \in \Omega \quad (7)$$

then inequality (1) is satisfied for all $u \in \mathcal{B}(B)$.

Proof: Since the expression in (3) is identically zero, inequality (1) is equivalent to

$$\oint_{\partial\Omega} \begin{bmatrix} (D^{\alpha-1}u)^T (M_{b1} + H_1) D^{\alpha-1}u \\ (D^{\alpha-1}u)^T (M_{b2} + H_2) D^{\alpha-1}u \end{bmatrix} \cdot \mathrm{d}\ell + \int_\Omega \left((D^\alpha u)^T \left(M_i - \bar{H}\right)(D^\alpha u)\right) \mathrm{d}\Omega \geq 0. \quad (8)$$

If (6) holds, then the line integral inequality in (8) holds for all $u \in \mathcal{B}$. If (7) holds, then the surface integral in (8) holds for all $u \in \mathcal{C}^\alpha(\Omega)$. Hence, if both (6) and (7) are satisfied, then (8) holds for all $u \in \mathcal{B}$, which implies that (1) holds for all $u \in \mathcal{B}$, since (1) and (8) are equivalent. ■

The terms introduced in the integrand by matrices H_1, H_2 do not affect the value of the integral and allow for a test for positivity based on the positivity of the matrices in the quadratic representation of the surface and of the line integrals. With the solution to (6), it is possible to verify inequalities in subspaces as in (2), incorporating boundary values of the dependent variables.

Remark 1: Inequality (7) is a differential matrix inequality since $\bar{H}(x,y)$ is composed of continuously differentiable functions and their derivatives. ⋆

Remark 2: Green's Theorem exposes the non-uniqueness of the integral expression associated to the differential relations of the elements in vector $D^\alpha u$. ⋆

III. Square domain

The case of a square domain Ω in (1), see Figure 1, is detailed considering a particular form of the boundary conditions (2). In order to formulate conditions to verify the positivity of the line integral in (6), we convert it into a one-dimensional integral on the interval $[0,1]$.

Consider the domain $\Omega = [0,1] \times [0,1]$, and the set

$$\mathcal{B}(B) := \left\{ u(x,y) \in \mathcal{C}^\alpha(\Omega) \middle| B \begin{bmatrix} D^{\alpha-1}u(x,0) \\ D^{\alpha-1}u(1,y) \\ D^{\alpha-1}u(x,1) \\ D^{\alpha-1}u(0,y) \end{bmatrix} = 0 \right\}, \quad (9)$$

with $B \in \mathbb{R}^{n_b \times 4n\sigma(2,\alpha-1)}$, which corresponds to (2) defined with a piece-wise linear function $f(x,y)$ and to $g(x,y) = 0$.

Define $\bar{M}_{b1}(x,y) := M_{b1}(x,y) + H_1(x,y)$ and $\bar{M}_{b2}(x,y) := M_{b2}(x,y) + H_2(x,y)$. The line integral in (6)

$$\left[\partial_x\left((D^1u(x,y))^T H_2(x,y)D^1u(x,y)\right)-\partial_y\left((D^1u(x,y))^T H_1(x,y)D^1u(x,y)\right)\right]=(D^\alpha u)^T\bar{H}\left(D^\alpha u\right)$$
$$=\left(D^2u(x,y)\right)^T\left(\begin{bmatrix}-\frac{\partial}{\partial y}H_1(x,y)+\frac{\partial}{\partial x}H_2(x,y) & 0_{3\times 3}\\ 0_{3\times 3} & 0_{3\times 3}\end{bmatrix}\right.$$
$$\left.+He\left(\begin{bmatrix}-H_1(x,y)\\ 0_{3\times 3}\end{bmatrix}\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}+\begin{bmatrix}H_2(x,y)\\ 0_{3\times 3}\end{bmatrix}\begin{bmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix}\right)\right)D^2u(x,y) \quad (5)$$

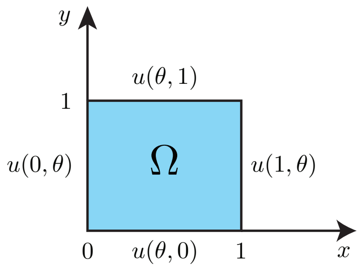

Fig. 1. Square domain Ω and boundary variables.

satisfies

$$\oint_{\partial\Omega}\begin{bmatrix}\left(D^{\alpha-1}u\right)^T\left(\bar{M}_{b1}(x,y)\right)D^{\alpha-1}u\\ \left(D^{\alpha-1}u\right)^T\left(\bar{M}_{b2}(x,y)\right)D^{\alpha-1}u\end{bmatrix}\cdot \mathrm{d}\ell$$
$$=\int_0^1\left((D^{\alpha-1}u(\theta,0))^T\bar{M}_{b1}(\theta,0)D^{\alpha-1}u(\theta,0)\right.$$
$$-(D^{\alpha-1}u(\theta,1))^T\bar{M}_{b1}(\theta,1)D^{\alpha-1}u(\theta,1)$$
$$+(D^{\alpha-1}u(1,\theta))^T\bar{M}_{b2}(1,\theta)D^{\alpha-1}u(1,\theta)$$
$$\left.-(D^{\alpha-1}u(0,\theta))^T\bar{M}_{b2}(0,\theta)D^{\alpha-1}u(0,\theta)\right)\ \mathrm{d}\theta. \quad (10)$$

To simplify the above expression, we introduce *boundary variables* $w_{ij}(\theta)$, $i=1,\ldots,4$, $j=0,\ldots,\alpha-1$, as

$$\begin{aligned} w_{1j}(\theta) &:= \partial_y^j u(\theta,0)\\ w_{2j}(\theta) &:= \partial_x^j u(1,\theta)\\ w_{3j}(\theta) &:= \partial_y^j u(\theta,1)\\ w_{4j}(\theta) &:= \partial_x^j u(0,\theta)\end{aligned} \quad (11)$$

and define

$$\tilde{w}_k(\theta) := \begin{bmatrix} D^{\alpha-1}w_{k0}(\theta)\\ D^{\alpha-2}w_{k1}(\theta)\\ \vdots\\ D^1 w_{k(\alpha-2)}(\theta)\\ w_{k(\alpha-1)}(\theta)\end{bmatrix}, \quad (12)$$

$k=1,\ldots,4$. These boundary variables satisfy

$$\begin{aligned}\tilde{w}_1(\theta) &= R_x D^{\alpha-1}u(\theta,0)\\ \tilde{w}_2(\theta) &= R_y D^{\alpha-1}u(1,\theta)\\ \tilde{w}_3(\theta) &= R_x D^{\alpha-1}u(\theta,1)\\ \tilde{w}_4(\theta) &= R_y D^{\alpha-1}u(0,\theta),\end{aligned} \quad (13)$$

where $R_x \in \mathbb{R}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha-1)}$ and $R_y \in \mathbb{R}^{n\sigma(2,\alpha-1)\times n\sigma(2,\alpha-1)}$ are permutation matrices.

The following example illustrates the above definitions

Example 1: Consider $n=2$ and $\alpha=3$ which gives

$$D^{\alpha-1}u=\begin{bmatrix} u^T & \partial_x u^T & \partial_y u^T & \partial_x^2 u^T & \partial_{xy}u^T & \partial_y^2 u^T\end{bmatrix}^T$$

we have

$$\begin{aligned}
&w_{10}(\theta)=u(\theta,0),\ w_{11}(\theta)=\partial_y u(\theta,0),\ w_{12}(\theta)=\partial_y^2 u(\theta,0);\\
&w_{20}(\theta)=u(1,\theta),\ w_{21}(\theta)=\partial_x u(1,\theta),\ w_{22}(\theta)=\partial_x^2 u(1,\theta);\\
&w_{30}(\theta)=u(\theta,1),\ w_{31}(\theta)=\partial_y u(\theta,1),\ w_{32}(\theta)=\partial_y^2 u(\theta,1);\\
&w_{40}(\theta)=u(0,\theta),\ w_{41}(\theta)=\partial_x u(0,\theta),\ w_{42}(\theta)=\partial_x^2 u(0,\theta);
\end{aligned}$$

$$\tilde{w}_1(\theta)=\begin{bmatrix}D^2 w_{10}(\theta)\\ D^1 w_{11}(\theta)\\ w_{12}(\theta)\end{bmatrix}=R_x D^2 u(\theta,0)$$

$$\tilde{w}_2(\theta)=\begin{bmatrix}D^2 w_{20}(\theta)\\ D^1 w_{21}(\theta)\\ w_{22}(\theta)\end{bmatrix}=R_y D^2 u(1,\theta)$$

(similar definitions follow for $\tilde{w}_3(\theta)$ and $\tilde{w}_4(\theta)$), with

$$R_x=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\quad R_y=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}.$$

•

The values of u at the vertices of the domain Ω establish the following relation for the values of $\tilde{w}_i$ at $\theta=0$, $\theta=1$

$$\begin{aligned}R_x^T\tilde{w}_1(1)-R_y^T\tilde{w}_2(0) &= 0\\ R_y^T\tilde{w}_2(1)-R_x^T\tilde{w}_3(1) &= 0\\ R_x^T\tilde{w}_3(0)-R_y^T\tilde{w}_4(1) &= 0\\ R_y^T\tilde{w}_4(0)-R_x^T\tilde{w}_1(0) &= 0,\end{aligned}$$

where we have used the fact that R_x and R_y satisfy $R_x^TR_x=R_y^TR_y=I$ (since these are permutation matrices). The above expression can be written in the compact form

$$\bar{B}\begin{bmatrix}\bar{w}(1)\\ \bar{w}(0)\end{bmatrix}=0 \quad (14)$$

with

$$\bar{w}(\theta):=\begin{bmatrix}\tilde{w}_1(\theta)\\ \tilde{w}_2(\theta)\\ \tilde{w}_3(\theta)\\ \tilde{w}_4(\theta)\end{bmatrix},$$

$$\bar{B}:=\begin{bmatrix}R_x^T&0&0&0&0&-R_y^T&0&0\\ 0&R_y^T&-R_x^T&0&0&0&0&0\\ 0&0&0&-R_y^T&0&0&R_x^T&0\\ 0&0&0&0&-R_x^T&0&0&R_y^T\end{bmatrix}.$$

Finally, define

$$\tilde{B} \quad := \quad B\left(Diag(R_x, R_y, R_x, R_y)\right).$$

and

$$\tilde{M}_b(\theta) := Diag\left(\tilde{M}_{b1}(\theta), \tilde{M}_{b2}(\theta), -\tilde{M}_{b3}(\theta), -\tilde{M}_{b4}(\theta)\right)$$

with

$$\begin{aligned}\tilde{M}_{b1}(\theta) &:= R_x \bar{M}_{b1}(\theta, 0) R_x^T \\ \tilde{M}_{b2}(\theta) &:= R_y \bar{M}_{b2}(1, \theta) R_y^T \\ \tilde{M}_{b3}(\theta) &:= R_x \bar{M}_{b1}(\theta, 1) R_x^T \\ \tilde{M}_{b4}(\theta) &:= R_y \bar{M}_{b2}(0, \theta) R_y^T,\end{aligned}$$

to allow for a compact representation of (9) and (10).

The following proposition states the equivalence between verifying (6) and checking an integral in one dimension.

Proposition 1: The following are equivalent

1) Inequality (6), with $\mathcal{B}(B)$ as in (9) holds.
2) Inequality

$$\int_0^1 (\bar{w}(\theta))^T \tilde{M}_b(\theta) \bar{w}(\theta) \mathrm{d}\theta \geq 0 \tag{15a}$$

holds for all w_{ij} in the set

$$\mathcal{B}(\tilde{B}, \bar{B}) := \left\{ w_{ij} \middle| \tilde{B}\bar{w}(\theta) = 0, \bar{B}\begin{bmatrix} \bar{w}(1) \\ \bar{w}(0) \end{bmatrix} = 0 \right\}. \tag{15b}$$

The above result simply reformulates the problem of verifying (6) into an expression which is amenable to the method to solve one-dimensional integral inequalities presented in [12, Proposition 1].

IV. Stability Analysis for Partial Differential Equations

This section introduces integral inequalities for the convergence of norms of the solutions of linear PDEs. We detail the formulation presented in [13]. Similar formulations are also found in [9], [14] and [15].

Definition 1: A nonlinear semi-group on a compact normed space C is a family of maps $\{S(t) \mid C \to C,\ t \geq t_0\}$ such that

- for each $t \geq t_0$, $S(t)$ is continuous from C to C,
- for each $u \in C$, the mapping $t \to S(t)u$ is continuous,
- $S(0)$ is the identity on C,
- $S(t)(S(\tau)u) = S(t+\tau)u$ for all $u \in C$ and all $t, \tau \geq 0$.

Definition 2: Let $\{S(t),\ t \geq t_0\}$ be a nonlinear semi-group on C and for any $u \in C$, let $Y(u) = \{S(t)u,\ t \geq t_0\}$ be the *orbit* through u. We say u is an equilibrium point if $Y(u) = \{u\}$.
An orbit $Y(u)$ is *stable* if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $t \geq t_0$, $\|S(t)u - S(t)v\|_C < \epsilon$ whenever $\|u - v\|_C < \delta(\epsilon)$, $v \in C$, where $\|\cdot\|_C$ is the norm defined on C. An orbit is *uniformly asymptotically stable* if it is stable and also there is a neighbourhood $D = \{v \in C \mid \|u - v\|_C < r\}$ such that $\|S(t)u - S(t)v\|_C \to 0$ as $t \to \infty$, uniformly for $v \in D$ [1]. Similarly, it is *exponentially stable* if there exist $\sigma, \gamma > 0$ such that

$$\|S(t)u - S(t)v\|_C \leq \gamma \|u - v\|_C e^{-\sigma(t - t_0)},$$

for all $t \geq t_0$ and all $u, v \in C$.

Definition 3: Let $\{S(t),\ t \geq t_0\}$ be a nonlinear semi-group on C. A Lyapunov function is a continuous real-valued function V on C such that

$$\partial_t V(u) = \lim_{t \to 0^+} \frac{V(S(t)u) - V(u)}{t} \leq 0, \tag{16}$$

for all $u \in C$.

Theorem 2: (Lyapunov Theorem for Nonlinear Semi-groups, [13, Theorem 4.1.4]), Let $\{S(t),\ t \geq t_0\}$ be a nonlinear semi-group, and let 0 be an equilibrium point in C. Suppose V is a Lyapunov function on C which satisfies $V(0) = 0$, and $V(u) \geq \alpha_1 \|u\|_C$ for $\alpha_1 > 0$ and $u \in C$. Then, 0 is stable. In addition, if $\partial_t V(u) \leq -\alpha_2 \|u\|_C$ for $\alpha_2 > 0$, then 0 is uniformly asymptotically stable.
For the proof of the above theorem, refer to [13, p.84]. The exponential stability of linear semi-groups can also be certified by the solution to the Lyapunov equation presented in [16, Theorem 5.1.3].

In what follows, we present the class of PDE systems and Lyapunov functionals studied in this paper. Consider the following well posed PDE system

$$\partial_t u = F((x, y), D^\alpha u), \tag{17}$$

$u(t_0, (x, y)) = u_0(x, y) \in \mathcal{M} \subset H^q(\Omega)$, where $q \in \mathbb{N}_0$. Let $F(x, D^\alpha u) =$, where $\mathcal{A}$ is a linear operator defined on $\mathcal{M}$, a closed subset of $H^q(\Omega)$. Continuous solutions to the PDE (17) exist in $\mathcal{M}$ and are unique, provided $\mathcal{A}$ generates a linear semi-group of contractions.

Theorem 3: Suppose there exist a function $V \in \mathcal{C}^1$, with $V(0) = 0$, and scalars c_1, c_2, $c_3 \in \mathbb{R}_{>0}$ such that

$$c_1 \|u\|_q^2 \leq V(u) \leq c_2 \|u\|_q^2 \tag{18}$$

$$\partial_t V(u) \leq -c_3 \|u\|_q^2 \tag{19}$$

then, the H^q norm of the trajectories of (17) satisfy

$$\|u(t, x, y)\|_q^2 \leq \frac{c_2}{c_1} \|u_0(x, y)\|_q^2 e^{-\frac{c_3}{c_1}(t - t_0)} \tag{20}$$

where $u_0 = u(t_0, x)$.

Proof: From (18)-(19), one has $\frac{\frac{\mathrm{d}V(u)}{\mathrm{d}t}}{V(u)} \leq -\frac{c_3}{c_1}$. Since $\frac{\frac{\mathrm{d}V(u)}{\mathrm{d}t}}{V(u)} = \frac{\mathrm{d}(\ln(V(u)))}{\mathrm{d}t}$, the integral in time of the above expression over $[t_0, t]$ yields $\ln(V(u(t))) - \ln(V(u(t_0))) \leq -\frac{c_3}{c_1}(t - t_0)$ which gives $V(u(t)) \leq V(u(t_0)) e^{-\frac{c_3}{c_1}(t - t_0)}$ Finally (20) is obtained by applying the bounds of (18) on the above inequality. ∎

Consider candidate Lyapunov functionals of the form

$$V(u) = \frac{1}{2} \int_\Omega (D^q u)^T P(x, y) (D^q u)\, \mathrm{d}\Omega,\ P(x, y) > 0 \quad \forall (x, y) \in \Omega. \tag{21}$$

[1] i.e. $\forall \varepsilon > 0, \exists T > 0 :\ t > T$ s.t. $\|S(t)u - S(t)v\|_C < \varepsilon,\ \forall v \in D$.

That is, $V(u) = \frac{1}{2}\|u\|^2_{(q,P(x,y))}$, the squared $P(x,y)$-weighted H^q-norm.

The following lemma states the equivalence of the weighted norm and the H^q-norm. Its proof is straightforward and therefore, omitted.

Lemma 2: If $P(x,y) > 0 \quad \forall x \in \Omega$, then the norms $\|u\|_{(q,P)}$ and $\|u\|_q$ are equivalent.

Remark 3: For $q_1 < q_2$, the space H^{q_1} is embedded in H^{q_2} [17, Sec 5.6]. Therefore, stability in H^{q_2}-norm implies stability in H^{q_1}-norm, but the converse does not hold. ⋆

Proposition 2: If there exists a function $P(x,y)$ and positive scalars ϵ_1, ϵ_2 such that

$$\int_\Omega \left[(D^q u)^T P(x,y)(D^q u) - \epsilon_1 (D^q u)^T (D^q u)\right] \mathrm{d}\Omega \geq 0 \tag{22a}$$

$$-\int_\Omega \left[2(D^q u)^T P(x,y) F((x,y)D^\alpha u) + \epsilon_2 (D^q u)^T (D^q u)\right] \mathrm{d}\Omega \geq 0 \tag{22b}$$

then the H^q norm of solutions to (17) satisfy (20) with $c_1 = \min_\Omega(\lambda_{min}(P(x,y)))$, $c_2 = \max_\Omega(\lambda_{max}(P(x,y)))$, and $c_3 = \epsilon_2$.

Inequalities (22a)-(22b) are integral inequalities such as the ones studied in Section II. The sets $\mathcal{B}(B)$ as in (2) associated to the inequalities are defined by the domain of the PDE operators. The results of Sections II can therefore be applied to (22a)-(22b) whenever the integrand is a polynomial on the dependent variables.

V. Examples

This section presents a convex optimization formulation to solve (7) and applies the formulation to solve one-dimensional integral inequality presented in [12], [18] to solve (15).

A. Semidefinite programming formulation

Whenever matrix $M_i(x,y)$ is a polynomial on variables x and y and we impose polynomial dependence of $H_k(x,y)$, $k \in \{1,2\}$ on variables x and y, the matrix inequality in (7) becomes a polynomial matrix inequality. Note that the set $\Omega = [0,1] \times [0,1]$, is a semi-algebraic set as $\Omega = \{(x,y) \in \mathbb{R}^2 | \omega(x) \geq 0, \omega(y) \geq 0\}$ with $\omega(\xi) := \xi(1-\xi)$. A SOS formulation to (7) is then obtained from a straightforward application of Putinar's Positivstellensatz [19, Theorem 2.14] as in the corollary below.

Corollary 1: For $\left(M_i(x,y) - \bar{H}(x,y)\right) \in \mathcal{R}[x,y]$, if there exists $N_x(x,y)$, $N_y(x,y) \in \Sigma^{n\sigma(2,\alpha)\times n\sigma(2,\alpha)}[x,y]$ such that

$$\left(M_i(x,y) - \bar{H}(x,y)\right) - N_x(x,y)\omega(x) - N_y(x,y)\omega(y) \in \Sigma^{n\sigma(2,\alpha)\times n\sigma(2,\alpha)}[x,y] \tag{23}$$

then (7) holds.

Remark 4: Whenever the coefficients of $M_i(x,y)$ and $\bar{H}(x,y)$ depend affinely in unknown parameters and the degrees of $N_x(x,y)$ and $N_y(x,y)$ are fixed, checking whether (23) holds can be cast as a feasibility problem of a convex set of constraints, an SDP, whose dimensions (number of constraints and decision variables) depend on the degree of $M_i(x,y) - \bar{H}(x,y)$, the degrees of N_x and N_y in x, y and on the dimension of matrix $M_i(x,y) - \bar{H}(x,y)$, given by $n\sigma(2,\alpha)$. ⋆

B. Numerical examples

We formulate the Lyapunov stability conditions for the exponential decay of the $\mathcal{L}^2$-norm of the heat equation. The test with such a simple linear PDE allows us to compare the decay bounds obtained with the eigenvalue analysis of the diffusion operator in the squared domain. We then solve the SOS constraints obtained by applying Corollary 1 to inequalities (22) (the constraints are cast as SDP with SOSTOOLS [20] and solved with SeDuMi [21]).

The heat equation - Dirichlet Boundary conditions

$$\partial_t u(t,x,y) = \partial_x^2 u(t,x,y) + \partial_y^2 u(t,x,y) \tag{24a}$$

$$(x,y) \in \Omega = [0,1] \times [0,1], \ t > 0 \tag{24b}$$

$$u(t,0,y) = u(t,1,y) = u(t,x,0) = u(t,x,1) = 0 \tag{24c}$$

The above boundary conditions define the set $\mathcal{B}$ in (9) with

$$B = \left[\begin{array}{c|c|c|c} B_1 & 0 & 0 & 0 \\ \hline 0 & B_2 & 0 & 0 \\ \hline 0 & 0 & B_3 & 0 \\ \hline 0 & 0 & 0 & B_4 \end{array}\right] \tag{25}$$

where $B_1 = B_2 = B_3 = B_4 = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]$.

The inequalities (22) with polynomial weighting function $p(x,y)$, are defined with $q = 0$ (giving $D^q u = u$) and $\epsilon_2 = \lambda$, are respectively given by

$$\int_\Omega \left[(p(x,y) - \epsilon_1)u^2\right] \mathrm{d}\Omega \geq 0 \tag{26a}$$

$$-\int_\Omega 2\left[\lambda p(x,y)u^2 + p(x,y)u(\partial_x^2 u + \partial_y^2 u)\right] \mathrm{d}\Omega \geq 0. \tag{26b}$$

The application of Theorem 1 and the SOS formulation of Corollary 1 give, for the surface term (7) of (26a)

$$p(x,y) - n_x(x,y)\omega(x) - n_y(x,y)\omega(y) \in \Sigma[x,y]$$

$$n_x(x,y) \in \Sigma[x,y], \ n_y(x,y) \in \Sigma[x,y]$$

and (28) for (26b). The SOS constraints related to the boundary inequality (15), are obtained as detailed in [12].

The numerical results provide polynomial Lyapunov certificates for the $\mathcal{L}^2$ stability of the solutions of (24). We have considered weighting functionals of different degrees in the Lyapunov function $p(x,y)$ and have imposed the same degree for the terms introduced by Green's Theorem. Table I lists the obtained values.

TABLE I

Decay bound estimates obtained with different degrees of polynomial functions $P(x,y)$, $H_i(x,y)$, $i = 1,2$

$deg(P(x,y)), deg(H_i(x,y))$	3	4	5	6	7
$\lambda^*/2\pi^2$	0.58	0.68	0.75	0.79	0.82

$$
\begin{bmatrix} -\lambda p(x,y) & 0 & 0 & p(x,y) & 0 & p(x,y) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p(x,y) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p(x,y) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \bar{H}(x,y) - N_x(x,y)\omega(x) - N_y(x,y)\omega(y) \in \Sigma^{\sigma(2,2)\times\sigma(2,2)}[x,y] \tag{28a}
$$

$$
N_x(x,y), N_y(x,y) \in \Sigma^{\sigma(2,2)\times\sigma(2,2)}[x,y] \tag{28b}
$$

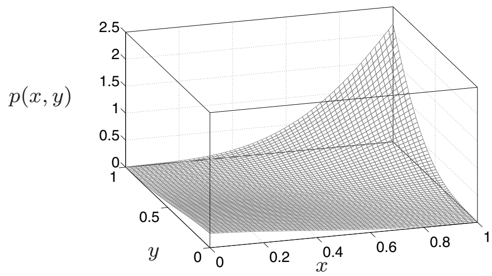

Fig. 2. Weighting function $p(x,y)$ of Lyapunov function $V(u(x,y))$.

VI. Conclusion

We proposed tests to verify the positivity of two-dimensional integral expressions with quadratic integrands. Green's Theorem is applied to obtain a set of quadratic representations of a given integral inequality. Then, the positivity of the surface integral is studied by analysing the positivity of the matrices in the quadratic representations. An important feature of the proposed approach on how it allows for checking the inequalities for sets defined by the boundary values of the dependent variables. We detail the method for unit square domains. The results generalise the approach for one-dimensional integral inequalities with quadratic integrands presented in [12] and will be extended to solve dissipation inequalities as in [22] and to safety verification as in [23].

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